《数学史》论文

学年学期: 2025-2026-2

论文编号: 2025-2026-2-

提交日期: 2026 年 4 月 30 日

姓　　名: 陈潇扬

院　　系: 求真书院

班　　级: 32

学　　号: 2023013271

评阅意见:
论文成绩（百分制）:
论文考核意见: ☐ 通过 ☐ 不通过

Higher Lie algebras and related topics

Xiaoyang Chen
2026 Spring
complied on 2026.04.30

Contents

1. Introduction

This essay will give a brief survey of the history of higher Lie algebras and related mathematical problems. Higher Lie algebras, as its name suggests, is a generalization of Lie algebras to higher categories. The first example of a higher Lie algebra is called differential graded Lie algebras (which we will write DGLA for short). It lives in the derived category of chain complexes, which is possibly one of the earliest higher categories known to us. As we shall see, DGLA was possibly introduced first by Quillen in his paper "Rational Homotopy Theory" to classify the rational homotopy types of simply connected spaces by the DGLA structure of its homotopy groups given by the Whitehead product. In this paper, the first idea of Koszul duality of Lie algebras was also introduced, namely firstly associating a DGLA to a rational homotopy type of a simply connected space, then to the coalgebra of its homology. This concept inspired the definition of another type of higher Lie algebras, which is known now by $\mathbb{L}_\infty$-algebras.

Besides in rational homotopy theory, later Deligne discovered that the DGLAs play an important role in deformation theory. Namely, we may construct DGLAs to solve deformation problems. This idea then inspired Kontsevich for his theory of mirror symmetry.

Before going into the topic of higher Lie algebras, we must first introduce the language of operads so as to formulate the definition of higher Lie algebras. To give a first intuition, operads is a method that can allow us to define algebras with operations at our own will. For the reader who is familiar with the concept of universal algebras, recall that we may define an universal algebra A with arbitrarily many operations $\left(f_n : A^{k_n} \to A\right)_{n\in\Sigma}$. This Σ parametrizing the operations serves the same role as an operad.

Operads are originally introduced as a tool for analysing iterated loopspaces in topology by May in the 1970s. His work, in short, proved that iterated loopspaces (e.g. $\Omega^n X$) are instances of "algebras of the $\mathbb{E}_n$ operad", which defines an equivalence of categories of algebras of certain operads with the subcategory of spaces which are iterated loopspaces. Here, operads were introduced to solve the problem of higher homotopical coherence of commutativity. For example, recall that H-spaces are spaces X equipped with a unital multiplication $e : * \to X$ and $m : X \times X \to X$. Now, if we wish to impose an associativity condition on this space X, we need an associator that remembers a globally defined path $a : (x \cdot y) \cdot z \to x \cdot (y \cdot z)$, where globally defined means that this path is continuously chosen with respect to x, y, and z (i.e. a homotopy between the two maps $X^3 \to X$). However, this gives an even further problem of associating 4 elements. If we add all n-ary associators, we would arrive at the so called $\mathbb{A}_\infty$-algebra. If we apply this idea to generalize the commutative algebras, we would obtain algebras coherently homotopically commutative to the n-th level, which is exactly the algebras of the $\mathbb{E}_n$ operad, or the $\mathbb{E}_\infty$ operad if the algebra is as commutative as possible.

This thought of including all homotopical data was later developed into the theory of ∞-operads, which played a central role in the theory of ∞-categories. It would be used in our definition of $\mathbb{L}_\infty$ algebras, since we will generally work in the category of chain complexes, which is an ∞-category.

In this essay, we will first give the necessary categorical preliminaries, define the $\mathbb{L}_\infty$-operad, and discuss its role in deformation theory.

2. Operads and Categorical Preliminaries

2.1. Symmetric sequences

Following [Chi05], we sketch the theory of operads in terms of symmtric sequences.

Definition 2.1.1. For a category $\mathcal{C}$, we define a symmetric sequence in $\mathcal{C}$ to be a sequence of elements $c_i \in \mathcal{C}$ for $i \geqslant 1$, such that c_i admits an Σ_i-action, or equivalently a functor $F : \mathbf{Fin}^{\simeq} \to \mathcal{C}$. This equivalence is given by $[i] \mapsto c_i$, where $[i]$ is the set with i elements. Here, we require $\mathbf{Fin}^{\simeq}$ to be the category of non-empty finite sets with bijections.

Construction 2.1.2. Let $\mathcal{C}$ be a pointed closed symmetric monoidal category. We define the composition product on $\mathrm{Fun}(\mathbf{Fin}^{\simeq}, \mathcal{C})$ by

$$F \circ G(A) = \coprod_{A = \coprod_{j \in J} A_j} F(J) \otimes \left(\bigotimes_{j \in J} G(A_j) \right)$$

This is a monoidal structure on $\mathrm{Fun}(\mathbf{Fin}^{\simeq}, \mathcal{C})$, with tensor unit given by the unit symmetric sequence given by $I([1]) = 1_{\mathcal{C}}$ and $I([n]) = *$, where $*$ is the null object of $\mathcal{C}$.

Definition 2.1.3. We define an operad in $\mathcal{C}$ to be a non-unital monoid in $\mathrm{Fun}(\mathbf{Fin}^{\simeq}, \mathcal{C})$.

Unwinding definitions, an operad is a symmetric sequence P with structure maps for every $a \in A$ an element of a finite set $\circ_a : P(A) \otimes P(B) \to P((A - \{a\}) \cup B)$, such that the following diagrams commute:

$$\begin{tikzcd}
P(A) \otimes P(B) \otimes P(C) \arrow[r] \arrow[d] & P(A) \otimes P((B - \{b\}) \cup C) \arrow[d] \\
P((A - \{a\}) \cup B) \otimes P(C) \arrow[r] & P((A - \{a\}) \cup (B - \{b\}) \cup C)
\end{tikzcd}$$

$$\begin{tikzcd}
P(A) \otimes P(B) \otimes P(C) \arrow[rr] \arrow[d] & & P((A - \{a\}) \cup B) \otimes P(C) \arrow[d] \\
P(A) \otimes P(C) \otimes P(B) \arrow[r] & P((A - \{a'\} \cup C) \otimes P(B) \arrow[r] & P((A - \{a, a'\}) \cup B \cup C)
\end{tikzcd}$$

We define a morphism of operads to be the obvious morphism of non-unital monoids in $\mathrm{Fun}(\mathbf{Fin}^{\simeq}, \mathcal{C})$.

remark 2.1.1. The intuition for the above definition is the following: $P([n]) \in \mathcal{C}$ denotes the moduli object of n-ary operations. The above $P(A) \otimes P(B) \to P((A - \{a\}) \cup B)$ denotes the process of first taking B elements, and use an B-ary operation to make it 1 element, and finally plug it in the place a along with the rest elements indexed by $A - \{a\}$.

Construction 2.1.4. For an element $X \in \mathcal{C}$, we define its endomorphism operad by

$$\mathcal{E}_X([n]) = \operatorname{Hom}(X^{\otimes n}, X)$$

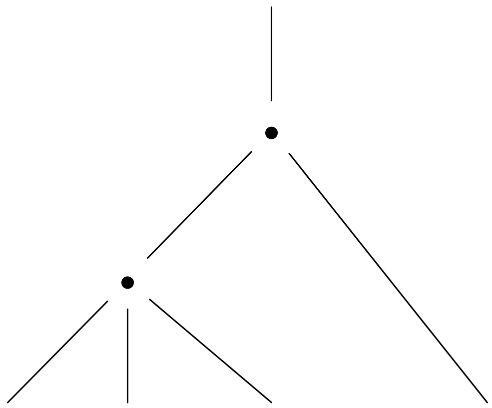
The Hom functor above is the internal Hom functor. The Σ_n-action on $\mathcal{E}_{X([n])}$ is given by the permutation action on $X^{\otimes n}$, and the structure maps are given by obvious compositions of morphisms.

Definition 2.1.5. We define an algebra over an operad P in the category $\mathcal{C}$ to be an object $A \in \mathcal{C}$ equipped with a map of operads $P \to \mathcal{E}_A$. Equivalently, this can be regarded as maps

$$P([n]) \otimes \bigotimes_{i \in [n]} A \to A$$

satisfying appropriate naturality, associativity, and unital axioms. We define the category of algebras to be $\mathbf{Alg}_P$.

The above analysis is often written in the language of trees, where we regard an n-ary operation as a tree with only 1 root and n-leaves, as shown in the following example with $A = [2]$, $a = 1$ and $B = [3]$:



This idea is now known as the theory of dendroidal sets, developed by Moerdijk and Weiss ([MW07]). We can view the effect of this theory to operads as simplicial sets to category theory. We then make the above idea more explicit. Here our definition of trees slightly differs from the most common convention for defining trees.

Definition 2.1.6. We define a tree T to be a structure consisting of two sets (V, E), where V denote a set of vertices, E denote an non-empty set of edges, a distinguished element $r \in E$ called the root, and finally the following maps.

(1) $I : E - \{r\} \to V$, which we think of assigning an edge e to the vertex which is the input of this edge.

(2) $O : V \to E$, which assigns a vertex to the edge which outputs to this vertex.

We define a leaf to be an edge in the complement of O, as depicted in the above picture (note there are no vertices below the leaf edges). An internal edge is an edge that is not a leaf.

We define the valence of a vertex v to be the number of edges such that $I(e) = v$.

We define an external vertex to be a vertex with exactly one internal edge connected to it. We note in particular that the root is not an internal edge.

We then define the category of trees and some useful subcategories.

Definition 2.1.7. We denote the category of trees by Ω. In this category, the object are trees, and the morphisms are generated by the following elementary morphisms:
(1) (internal faces) For a tree T and one of its internal edge e, there is a morphism $\partial_e : T/e \to T$, where T/e is the tree obtained from removing e and identifying its ends. We regard the new vertex as $I(e)$ rather than the other vertex v with $O(v) = e$.
(2) (external faces) For a tree T and an external vertex, there is a morphism $\partial_v : T/v \to T$, where T/v is the tree obtained from removing v and the leaves attached to it.
(3) (degeneracies) For a tree T and an edge e, there is a morphism $\sigma_e : \sigma_e T \to T$, where σ_e is obtained by splitting e into two edges e_1, e_2, separated by a new vertex.

remark 2.1.2. In the definition of external faces, if we remove an external vertex connected to a root, we define the new tree to be the one obtained by removing the root and the leaves attached to the vertex, and leaving the previous internal edge as the new root.

Definition 2.1.8. We define an external (internal) face to be a composition of elementary external (internal) faces. We also call the source of an external face by a subtree.

Definition 2.1.9. We define a tree to be open if there is no vertices of valence 0, and reduced if there is no vertices of valence 1. We define the category Ω_r^o to be the full subcategory of Ω spanned by open and reduced trees.

We define the category $\mathbb{A}$ to be the wide subcategory of Ω_r^o where the morphisms in $\mathbb{A}$ are compositions of isomorphisms and internal faces. We define the category $\mathbb{R}$ to be the wide subcategory of Ω_r^o where the morphisms in $\mathbb{R}$ are the morphisms preserving the root edge.

In this section, we assume every tree to be open and reduced.

Besides these, we have an additional operation on trees.

Definition 2.1.10. For trees S and R and a leaf e of S, we denote $T = S \circ_e R$ if T is obtained as a tree by identifying the root edge of R and e, called the grafting of S and R along e.

Note we can identify S and R as subtrees of T. S is obtained by cutting off all external vertices below e, and R is obtained by cutting off all external vertices not below e.

This operation is important since it generates all trees from some basic examples.

Definition 2.1.11. We define the n-corolla tree C_n by the tree with one root and n-leaves.

Lemma 2.1.12. All trees can be written as a iterated grafting of corolla trees.

Proof: We first prove that in a tree T that is not a corolla has an internal edge e connected to an external vertex. To prove this, since T is not a corolla, there is a leaf not adjacent to the root edge. Then $I(e)$ is the vertex we are searching for.

Then, we perform induction on the number of vertices of T. Notice by definition we have a tree with one vertex is always a corolla. Then, for a tree T with n vertices, we may cut T along the internal edge e that connects to an external vertex, so writing T as a grafting of a tree with $n-1$ vertices and a corolla. This finishes the induction. $\square$

Definition 2.1.13. We define a preoperad in a pointed closed symmetric monoidal category $\mathcal{C}$ to be a presheaf P over $\mathbb{A}$ in $\mathcal{C}$ equipped with a structure map

$$\theta_{S \circ_e T} : P(S \circ_e T) \to P(S) \otimes P(T)$$

It is called an operad if furthermore this structure map is an equivalence.

Definition 2.1.14. For a preoperad P in a category $\mathcal{C}$, we define ite prealgebra to be a presheaf $A : \mathbb{R}^{\mathrm{op}} \to \mathcal{C}$, and for each morphism $\alpha : S \to T$ in $\mathbb{R}$, a structure morphism

$$\tau^\alpha : A(T) \to P(S) \otimes A(T/S)$$

where $A(T/S) = \bigotimes_\ell A(T_\ell)$, where ℓ runs over the leaves of S, and T_ℓ is the subtree of T consisting of everything below $\alpha(\ell)$.

If P is an operad and the structure morphisms are all equivalences, then we define A to be an algebra of P.

Then we give a construction relating the two definitions.

Construction 2.1.15. Let P be a monoid in the category of symmetric sequence in $\mathcal{C}$. We define its nerve to be the following presheaf on $\mathbb{A}$

$$NP(T) := \bigotimes_v P(i(v))$$

where $i(v)$ denote the finite set of inputting edges to v.

We explicitly construct for a morphism $S \to T$ a morphism $NP(T) \to NP(S)$. Note that by definition, it suffices to construct the morphism for the case where we have $\partial_e : S \to T$ is an elementary internal face. In this case, since e is an internal edge, suppose we have $e = O(v_0)$ and $I(e) = v_1$, we have the following map

$$NP(T) = \bigotimes_{v \neq v_0, v_1} P(i(v)) \otimes (P(i(v_0)) \otimes P(i(v_1))) \to \bigotimes_{v \neq v_1} P(i(v)) \otimes P(i(v_1')) = NP(S)$$

given by the identity on the first factor, and on the second factor given by the structure map given previously $\circ_e : P(i(v_0)) \otimes P(i(v_1)) \to P(i(v_1'))$ (we note here that $i(v_1')$ is by definition given by $i(v_0) \cup i(v_1) - \{e\}$).

Theorem 2.1.16. The second definition of operads is equivalent to the category of operads with $P([1]) = 1_{\mathcal{C}}$, where $1_{\mathcal{C}}$ is the tensor unit.

Proof: We give an inverse to the above functor. We send an operad P to the symmetric sequence $\{P(C_n)\}$, where the Σ_n action on $P(C_n)$ is given by the isomorphism $C_n \to C_n$ permuting the n-leaves. Then, the required structure maps are exactly the structure maps given by grafting corolla trees. Then it is easy to verify that these two functors are inverses to each other on our given subcategories. □

Proposition 2.1.17. For a symmetric sequence P which is an operad and such that $P(1) = 1_{\mathcal{C}}$, the category of algebras of P is equivalent to the category of algebras of NP.

Proof: Again, the equivalence is given by restricting the presheaf to the corolla trees and extending the presheaf from the corolla trees to the whole category $\mathbb{R}$ via the isomorphism

$$A(T) \simeq P(S) \otimes A(T/S)$$

We omit the details. □

With the basic work done, we present some basic examples of operads and their algebras. For simplicity, we temporarily work in operads and algebras in the category $\mathbf{Vect}_k$.

Example 2.1.3. We define the operad **Comm** to be the operad with $\mathbf{Comm}([n]) = k$ for all n. Then, for A its algebra, by taking $[n] = 2$ we obtain a map $\mathbf{Comm}([2]) \otimes A^{\otimes 2} \to A$, which is also equivalent to a map $A \otimes A \to A$, which we can view as a multiplication map. Also note that under the Σ_2-action, the space **Comm**$([2])$ stays invariant, which means that we have a commutativity condition:

$$\begin{tikzcd} A \otimes A \arrow[r] \arrow[d, "\tau"'] & A \\ A \otimes A \arrow[ur] & \end{tikzcd}$$

where τ denote the map swapping the two factors.

Likewise, the space **Comm**$([3]) = k$ implies the associativity conditions, and **Comm**$([n]) = k$ implies that there are no other relations. Hence, the algebras for the operad **Comm** are the non-unital commutative k-algebras.

A more intuitive description of **Comm**$([n])$ to be the subspace of $k[x_1, \cdots, x_n]$ with each x_i occuring exactly once, which is obviously one dimensional.

Example 2.1.4. We define the operad **Assoc** to be the operad with **Assoc**$([n])$ be the subspace of $k\langle x_1, \cdots, x_n\rangle$ (this is the free associative algebra on n generators) spanned by elements that contain each x_i exactly once, i.e., spanned by the elements $\{x_1 \cdots x_n, \cdots, x_n \cdots x_1\}$. Explicitly, this is the standard Σ_n-representation. Then, one can easily verify that the algebras of this operad are the non-unital associative algebras.

Example 2.1.5. We define the operead **Lie** to be the operad with **Lie**$([n])$ be the subspace of $k\langle x_1, \cdots, x_n\rangle$ spanned by iterated Lie brackets containing each element once, for example, an element of the form

$$[[\cdots[[x_1, x_2], x_3], \cdots], x_n]$$

This might seem complicated at first, but it is an $(n-1)!$-dimensional Σ_n-representation $\mathrm{ind}_{C_n}^{\Sigma_n}(\chi)$, where χ is the representation of C_n corresponding to a primitive n-th root of unity. By definition, the algebras of this operad are the Lie algebras.

remark 2.1.6. It is slightly unnatural to consider these non-unital versions of **Assoc** and **Comm** at first, but the point is that they differs not much from the unital versions. To obtain the unital version, it suffices to change **Assoc**$([0])$ and **Comm**$([0])$ to k instead of 0 (the nullary operations of operads are simply constants in the algebra, such that the unit in associative algebras). However, in our conventions, the data of an operad does not contain information of degree 0, so we may just adjust ourselves to this setting.

Now we have basically finished the part on operads, but we still face a further possible generalization. We recall that in the above definition, we required $P([1]) = 1_{\mathcal{C}}$, but if we do not require this, our theory of operads would be slightly modified in to the theory of colored operads. Since our main focus, the $\mathbb{L}_\infty$ operad is indeed single colored, we will not talk too much about this. We will only give the definition, the intuition and some examples, so as to be better prepared for the ∞-operads in the next section.

Definition 2.1.18. Let C be a set of colors (this is just a set with no structures). We define a colored operad P in $\mathcal{C}$ with color C to be a family of objects in $\mathcal{C}$ along with some structural maps, indexed by the following. Given any n and colors $c_1, \cdots, c_n, c \in C$, an object $P(c_1, \cdots, c_n; c) \in \mathcal{C}$, to be thought as the space of operations that takes in the colors $c_1, \cdots, c_n$ and spits out a color c. Then, we require the following structure maps

(1) A unit $1_{\mathcal{C}} \to P(c; c)$ for each color c.
(2) For each permutation $\sigma \in \Sigma_n$, an isomorphism $\sigma^* : P(c_1, \cdots, c_n; c) \to P\left(c_{\sigma(1)}, \cdots, c_{\sigma(n)}; c\right)$ permuting the factors.
(3) For any sequence of colors $c_1, \cdots, c_n, c$ and any n-tuple of sequences $d_1^i, \cdots, d_{k_i}^i$, a composition map

$$m : P(c_1, \cdots, c_n; c) \otimes \bigotimes_{i=1}^{n} P\left(d_1^i, \cdots, d_{k_i}^i; c_i\right) \to P\left(d_1^1, \cdots, d_{k_1}^1, \cdots, d_1^n, \cdots, d_{k_n}^n; c\right)$$

We define a morphism of operads with color C to be the obvious one commuting with the above structure maps. We do not talk of morphism of operads with different colors.

Corollary 2.1.19. By definition, a single colored operad (i.e., $C = [1]$ the singleton set) is just an operad previously defined.

Construction 2.1.20. For C a set of colors $C = \{c_1, \cdots, c_m\}$ and objects $A_{c_1}, \cdots, A_{c_m} \in \mathcal{C}$, we define the colored endomorphism operad $\mathcal{E}\left(A_{c_1}, \cdots, A_{c_m}\right)$ to be the colored operad with

$$\mathcal{E}(c_1, \cdots, c_n; c) = \mathrm{Hom}\left(A_{c_1} \otimes \cdots \otimes A_{c_n}, A_c\right)$$

Again, the Hom functor above is the internal Hom functor, and all the structure maps are given by identities, permutations of factors, and compositions.

Definition 2.1.21. For P a colored operad in $\mathcal{C}$ with colors C, we define an algebra of P to be a family of objects indexed by C, say $A_{c_1}, \cdots, A_{c_m} \in \mathcal{C}$, with a morphism of colored operads

$$P \to \mathcal{E}\left(A_{c_1}, \cdots, A_{c_m}\right)$$

Example 2.1.7. We define the operad $\mathbf{Mod}_{\mathbf{Comm}}$ to be the colored operad with the set of colors $C = \{a, m\}$. We define $P(c_1, \cdots, c_n; c)$ to be

(1) If there are more than one $c_i = m$, $P(c_1, \cdots, c_n; c) = *$.
(2) If exactly one $c_i = m$ and $c = a$, $P(c_1, \cdots, c_n; c) = *$.
(3) If all $c_i = a$ and $c = m$, $P(c_1, \cdots, c_n; c) = *$
(4) If exactly one $c_i = m$ and $c = m$, $P(c_1, \cdots, c_n; c) = 1_{\mathcal{C}}$.
(5) If all $c_i = a$ and $c = a$, $P(c_1, \cdots, c_n; c) = 1_{\mathcal{C}}$.

The algebras of this operad, as its name suggests, consists of a pair (A, M) where A is a commutative algebra and M is an A-module, which is easily verified by definition.

Example 2.1.8. For any tree $T \in \Omega$, it is possible to define an colored operad freely generated by T, which we will abuse notation and denote it also by T. We define its colors to be the edges of T, and an operation $p \in T(c_1, \cdots, c_n; c)$ is simply a subtree of T with root c and leaves $c_1, \cdots, c_n$, with composition of operations given by grafting of trees.

This example provides a fully faithful embedding $\Omega \hookrightarrow \mathbf{Op}$ of trees into the category of operads.

remark 2.1.9. In fact, for any colored operad P, one may define the module operad for P given by $\mathbf{Mod}_P$, whose algebra is pairs (A, M) where A is an P-algebra and M is an A-module in a suitable sense. In particular, for P a single colored operad, whose algebras A resembles an object in $\mathcal{C}$ with some operations defined on it, its module M can really be thought as an object in $\mathcal{C}$ with an A-action. This intuition can be found by inspecting the module operads for **Comm**, **Assoc**, and **Lie**.

remark 2.1.10. Just as operads, a colored operad encodes the information of different types of objects with possible interactions. We should see in the next section that for ∞-operads, the space of colors could be much more exotic than merely being a set.

2.2. ∞-categories and ∞-operads

The above analysis applies to general 1-categories, but faces some problems in ∞-categories. In this section we explain the method to generalize operads to ∞-categories developed by Lurie, and the method of dendroidal sets developed by Moerdijk. For readers not familiar with ∞-categories, you may temporarily think it as a **Top**-enriched category where the mapping spaces are introduced to remember the homotopical data of maps.

We first introduce Lurie's approach, as introduced in [Lur17]. But before we really start, we must recall for the reader some basic concepts in the quasi-category model of ∞-categories introduced by Lurie in his book Higher topos theory ([Lur08]).

Definition 2.2.1. We define the category of simplices Δ to be the full subcategory of **Cat** spanned by the totally ordered finite posets, i.e. categories of the form

$$\Delta^n := \{c_0 \to c_1 \to \cdots \to c_n\}$$

Explicitly, Δ is the category with objects (possibly empty) finite totally ordered sets and morphisms the maps preserving the order.

We define a simplicial object in the category $\mathcal{C}$ to be a presheaf $F : \Delta^{\mathrm{op}} \to \mathcal{C}$. In particular a simplicial set is a presheaf $S : \Delta^{\mathrm{op}} \to \mathbf{Set}$. We will define $F_n := F(\Delta^n)$

Our quasi-categories would be modeled over simplicial sets.

Definition 2.2.2. We define the following simplicial sets.
(1) Let Δ^n denote the presheaf represented by the object Δ^n. This object looks like the topological n-simplex.

(2) Let $\partial\Delta^n \subset \Delta^n$ denote the subcomplex obtained by removing the n-simplex. This object looks like the topological n-simplex with its interior removed.
(3) Let $\Lambda_i^n \subset \Delta^n$ denote the subcomplex obtained by removing the n-simplex and the $(n-1)$-simplex facing the i-th vertex. This object looks like the horn in Δ^n facing the i-th vertex, and the inclusion above is often denoted by the horn inclusion.

Construction 2.2.3. We consider the following two families of maps in Δ:
(1) The face inclusions $d_i^n : \Delta^{n-1} \hookrightarrow \Delta^n$ whose image leaves only out $i \in \Delta^n$.
(2) The degeneracy map $s_i^n : \Delta^{n+1} \twoheadrightarrow \Delta^n$ which is surjective and $s_i^n(i) = s_i^n(i+1) = i$.

Note all morphisms in Δ are compositions of d_i^n's and s_i^n's. We have the following automorphism $\mathrm{op} : \Delta \to \Delta$ given by identity on objects and sending d_i^n to d_{n-i}^n and sending s_i^n to s_{n-i}^n.

Definition 2.2.4. We define the opposite of a simplicial set $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$ by the following simplicial set obtained by precomposition

$$X^{\mathrm{op}} : \Delta^{\mathrm{op}} \overset{(\mathrm{op})^{\mathrm{op}}}{\to} \Delta^{\mathrm{op}} \overset{X}{\to} \mathbf{Set}$$

Definition 2.2.5. For a morphism $f : X \to S$ of simplicial sets, we define it to be:
(1) A left fibration if it has the lifting properties with respect to the inclusions $\Lambda_i^n \hookrightarrow \Delta^n$, $0 \leqslant i < n$.
(2) A right fibration if it has the lifting properties for the inclusions $\Lambda_i^n \hookrightarrow \Delta^n$, $0 < i \leqslant n$.
(3) An inner fibration if it has the lifting properties for the inclusions $\Lambda_i^n \hookrightarrow \Delta^n$, $0 < i < n$.
(4) A Kan fibration if it has the lifting properties for the inclusions $\Lambda_i^n \hookrightarrow \Delta^n$, $0 \leqslant i \leqslant n$.
(5) A trivial fibration if it has the lifting properties for the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$

To define more useful fibrations, we need the concept of slice simplicial sets.

remark 2.2.1. Recall for a 1-category $\mathcal{C}$, we may define its slice over an object $x \in \mathcal{C}$ by defining $\mathcal{C}_{/x}$ to be the category with objects $y \to x$ and morphism maps $y \to z$ such that their structure maps to x commute. The slice of simplicial sets generalizes this construction to defining $X_{/f}$ for maps of simplicial sets $f : Y \to X$.

Definition 2.2.6. For simplicial sets S and S', we define their join $S \star S'$ to be the simplicial set

$$(S \star S')(J) = \coprod_{J = I \cup I', I < I'} S(I) \times S'(I')$$

where I, I', J are finite totally ordered sets, and $I < I'$ means that for any $i \in I$ and $i' \in I'$, we have $i < i'$.

By definition, this operation is not symmetric. Also by definition, we have $\Delta^n \star \Delta^m \simeq \Delta^{n+m-1}$.

remark 2.2.2. This construction is analogous to the topological join functor, that is, for spaces X and Y, we may define their join to be $X \star Y := X \times I \times Y / (* \times 0 \times Y \cup X \times 1 \times *)$ (here $* \times 0 \times Y$ means gluing all points of the form $(x, 0, y)$ and $(x, 0, y')$, and the other side analogously).

Definition 2.2.7. For a simplicial set K, we define the left cone over K by $K^{\triangleleft} := \Delta^0 \star K$. Dually the right cone over K is defined to be $K^{\triangleright} := K \star \Delta^0$.

Lemma 2.2.8. The functors $(-) \star S : \mathbf{Set}^{\Delta^{\mathrm{op}}} \to \mathbf{Set}^{\Delta^{\mathrm{op}}}_{S/}$ and $S \star (-) : \mathbf{Set}^{\Delta^{\mathrm{op}}} \to \mathbf{Set}^{\Delta^{\mathrm{op}}}_{S/}$ commutes with colimits.

Proof: Recall that limits and colimits in the presheaf category is pointwise. Thus the conclusion follows from the fact that in **Set**, colimits distribute over limits. □

Definition 2.2.9. We define the slice operation to be the right adjoint of joins. Explicitly, for any map $p : K \to S$ of simplicial sets, there is a simplicial set $S_{/p}$ with the following universal property

$$\mathrm{Hom}_{\mathbf{Set}^{\Delta^{\mathrm{op}}}}\left(Y, S_{/p}\right) = \mathrm{Hom}_{\mathbf{Set}^{\Delta^{\mathrm{op}}}_{K/}}(Y \star K, S)$$

Dually, we define $S_{p/}$ to have the universal property

$$\mathrm{Hom}_{\mathbf{Set}^{\Delta^{\mathrm{op}}}}\left(Y, S_{p/}\right) = \mathrm{Hom}_{\mathbf{Set}^{\Delta^{\mathrm{op}}}_{K/}}(K \star Y, S)$$

These adjoints exists by the 1-categorical adjoint functor.

remark 2.2.3. Intuitionally, $S_{/p}$ is the category of left cones in S over p. This can be seen by taking Y to be the singleton Δ^0, so we have

$$\mathrm{Hom}_{\mathbf{Set}^{\Delta^{\mathrm{op}}}}\left(\Delta^0, S_{/p}\right) = \mathrm{Hom}_{\mathbf{Set}^{\Delta^{\mathrm{op}}}_{K/}}(K^{\triangleleft}, S)$$

and $S_{p/}$ has the likewise intuition.

In fact, the above method give an explicit way of computing $S_{/p}$ and $S_{p/}$ by taking Y to be the simplices.

Definition 2.2.10. Let $p : X \to S$ be an inner fibration of simplicial sets. Let $f : x \to y$ be an edge in X. We say f is p-Cartesian if the induced map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

is a trivial Kan fibration.

Dually, we say f is p-coCartesian if f^{op} is p^{op}-Cartesian.

remark 2.2.4. Ignoring homotopical data, we can see p-Cartesian morphism admits the following universal property: given $x \to y$ mapping by p to $s \to t$, suppose $z \to y$ maps to $r \to t$, if there is a map $r \to s$ making the bottom triangle commute, then there is a map $z \to x$ making the top triangle commutes mapping to $r \to s$, as shown in the picture below

$$
\begin{tikzcd}
z \arrow[dr, dashed] \arrow[drr] \arrow[dd, mapsto] \arrow[ddr, phantom, "\Uparrow"] & & \\
& x \arrow[r] \arrow[dd, mapsto] & y \arrow[dd, mapsto] \\
r \arrow[dr] \arrow[drr] & & \\
& s \arrow[r] & t
\end{tikzcd}
$$

Dually, a p-coCartesian morphism has the following depicted universal property:

$$
\begin{tikzcd}
& & z \arrow[dd, mapsto] \\
x \arrow[urr] \arrow[r] \arrow[dd, mapsto] & y \arrow[ur, dashed] \arrow[dd, mapsto] \arrow[r, phantom, "\Uparrow"] & \\
& & r \\
s \arrow[urr] \arrow[r] & t \arrow[ur] &
\end{tikzcd}
$$

Definition 2.2.11. For a map $p : X \to S$ of simplicial sets, we define it to be a Cartesian fibration if p is an inner fibration, and for every edge $f : x \to y$ of S, there is a $\tilde{f} : \tilde{x} \to \tilde{y}$ mapping to f and is p-Cartesian. We say p is a coCartesian fibration if p^{op} is a Cartesian fibration.

Definition 2.2.12. An ∞-category is a simplicial set K such that $K \to *$ is an inner fibration.

A ∞-groupoid is an ∞-category where every morphism is invertible.

remark 2.2.5. The intuition of the definition above is that the objects of an ∞-category K are maps $\Delta^0 \to K$, and the n-morphisms are maps $\Delta^n \to K$. In particular, the composition of morphisms are witnessed by 2-morphisms, so by the lifting property given by the inner fibration condition, we may define the composition of two morphisms $g \circ f$ by first regarding g and f as the two edges in the horn Λ^2_1, and then use the filling condition to fill it to a map $\Delta^2 \to K$, and define the composition to be the third map.

Definition 2.2.13. We define **Ani** to be the ∞-category of topological spaces.

We omit the fact here that all the above definitions works well with ∞-categories instead of merely simplicial sets.

For later use, we must present the following structure theorem for Cartesian fibrations (and coCartesian fibrations, of course).

Theorem 2.2.14 (Straightening and unstraightening). We have the following equivalence of ∞-categories

$$\mathrm{Cart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_\infty)$$

where the left side denote the category of Cartesian fibrations over $\mathcal{C}$.

Definition 2.2.15. We define $\langle n \rangle \in \mathbf{Fin}_*$ to be the pointed set with n elements and 1 additional point. We define $\langle n \rangle^\circ$ to be the subset of n-elements removing the distinguished point.

A morphism $f : \langle m \rangle \to \langle n \rangle$ is inert if for each $i \in \langle n \rangle^\circ$, we have $f^{-1}(i)$ has exactly 1 element.

We denote $\rho^i : \langle n \rangle \to \langle 1 \rangle$ by the standard inert morphism by sending i to 1 and every other thing to the distinguished point.

We define a morphism $f : \langle m \rangle \to \langle n \rangle$ to be active if $f^{-1}(*) = *$, where $*$ is the distinguished point.

Definition 2.2.16 ([Lur17], Definition 2.1.1.10). An ∞-operad is a functor $p : \mathcal{O}^\otimes \to \mathbf{Fin}_*$ of ∞-categories which satisfies the following conditions.

(1) For every inert morphism $f : \langle m \rangle \to \langle n \rangle$ and every object $C \in \mathcal{O}^\otimes_{\langle m \rangle}$ (we denote this the fiber over $\langle m \rangle$), there exists a p-coCartesian morphism $\overline{f} : C \to C'$ in $\mathcal{O}^\otimes$ lifting f. In particular, f induces a functor $f_! : \mathcal{O}^\otimes_{\langle m \rangle} \to \mathcal{O}^\otimes_{\langle n \rangle}$.

(2) Let $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and $C' \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ be objects, let $f : \langle m\rangle \to \langle n\rangle$ be a morphism in $\mathbf{Fin}_*$, and let $\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C')$ be the union of those connected components of $\mathrm{Map}(\mathcal{O}^{\otimes})$ which lie over $f \in \mathrm{Hom}_{\mathbf{Fin}_*}(\langle m\rangle, \langle n\rangle)$. Choose p-coCartesian morphisms $C' \to C'_i$ lying over the inert morphisms $\rho^i : \langle n\rangle \to \langle 1\rangle$ for $1 \leqslant i \leqslant n$. Then the induced maps

$$\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{1\leqslant i\leqslant n} \mathrm{Map}^{\rho^i\circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$

is a homotopy equivalence.

(3) For every finite collection of objects $C_1, \cdots, C_n \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$, there exists an object $C \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ and a collection of p-coCartesian morphisms $C \to C_i$ covering $\rho^i : \langle n\rangle \to \langle 1\rangle$.

We call $\mathcal{O} := \mathcal{O}^{\otimes}_{\langle 1\rangle}$ the colors of this operad.

remark 2.2.6. We present the basic intuitions for these axioms. Firstly, as we have defined, $\mathcal{O}$ serves as a generalization of colors of a colored operad. Then the three axioms ensures that for objects $C_1, \cdots, C_n \in \mathcal{O}$, they can be combined into "$C_1 \otimes \cdots \otimes C_n$", which lies over $\langle n\rangle$, and every object over $\langle n\rangle$ is of this form (explicitly $\mathcal{O}^{\otimes}_{\langle n\rangle} \simeq \mathcal{O}^n$). The second axiom furthermore ensures that for a map of objects $C \to D = D_1 \otimes \cdots \otimes D_n$ for $D_j \in \mathcal{O}$, this map must look like a tensor product of maps $C \to D_i$.

Example 2.2.7. The identity functor $\mathbf{Fin}_* \to \mathbf{Fin}_*$ is clearly an ∞-operad, which we denote by $\mathbf{Comm}^{\otimes}$. This is the ∞-analogue of the previously defined $\mathbf{Comm}$.

In the ∞-categorical setting, the definition of (symmetric) monoidal categories are much more difficult and subtle than in the classical case. The problem This difficulty leads to the following extremely important observation of realizing a symmetric monoidal category as an ∞-operad. We will explain more after the definition.

Definition 2.2.17. Let $p : \mathcal{O}^{\otimes} \to \mathbf{Fin}_*$ be an ∞-operad. We define a morphism f in $\mathcal{O}^{\otimes}$ to be inert if $p(f)$ is inert and f is p-coCartesian. We define a morphism f in $\mathcal{O}^{\otimes}$ to be active if $p(f)$ is active.

Definition 2.2.18. Let $p : \mathcal{O}^{\otimes} \to \mathbf{Fin}_*$ and $p' : \mathcal{O}'^{\otimes} \to \mathbf{Fin}_*$ be ∞-operads. An ∞-operad map $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ is a map of simplicial sets such that $p = p'f$, and f carries inert morphisms of $\mathcal{O}^{\otimes}$ to the inert morphisms of $\mathcal{O}'^{\otimes}$.

We denote $\mathbf{Alg}_{\mathcal{O}}(\mathcal{O}') \subset \mathbf{Fun}_{\mathbf{Fin}_*}(\mathcal{O}, \mathcal{O}')$ to be the full subcategory spanned by the ∞-operad maps.

Definition 2.2.19. We define a coCartesian fibration of operads to be a map of operads that is also a coCartesian fibration.

Definition 2.2.20. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. We define an $\mathcal{O}^{\otimes}$-monoidal ∞-category to be a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$.

Definition 2.2.21. We define a symmetric monoidal ∞-category to be a $\mathbf{Comm}^{\otimes}$-monoidal ∞-category.

remark 2.2.8. Let $\mathcal{C}$ be a symmetric monoidal ∞-cateegory. Then since $\mathcal{C} \to \mathbf{Fin}_*$ is a coCartesian fibration, consider the map $\langle 0 \rangle \to \langle 1 \rangle$ and the active map $\langle 2 \rangle \to \langle 1 \rangle$. Since $\mathcal{C} \to \mathbf{Fin}_*$ is a coCartesian map, by the straightening and unstraightening theorem they induce functors of categories $\mathcal{C}^{\otimes}_{\langle 2 \rangle} \simeq \mathcal{C}^2 \to \mathcal{C}$ and $\mathcal{C}^{\otimes}_{\langle 0 \rangle} \simeq \Delta^0 \to \mathcal{C}$, they may be respectively regarded as the tensor product and unit in $\mathcal{C}$.

Definition 2.2.22. Let $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads, and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}^{\otimes}$-monoidal category. We define the category of algebras $\mathbf{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ by the ∞-operad maps $\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ over $\mathcal{O}^{\otimes}$.

remark 2.2.9. This definition has the same intuition as algebras for classical operads, where we regard $\mathcal{C}^{\otimes}$ as some sort of endomorphism operad.

Again, we present some examples of ∞-operads.

Example 2.2.10. We may define the operad $\mathbb{E}_k^{\otimes}$ as follows:
(1) The objects of $\mathbb{E}_k^{\otimes}$ are the objects $\langle n \rangle$ as in $\mathbf{Fin}_*$.
(2) Given a pair of objects $\langle m \rangle, \langle n \rangle$, the space of morphisms between them are given by the following

$$\mathrm{Hom}_{\mathbb{E}_k^{\otimes}}(\langle m \rangle, \langle n \rangle) = \coprod_{f:\langle m \rangle \to \langle n \rangle} \prod_{1 \leqslant j \leqslant n} \mathrm{Rect}\big(\square^k \times f^{-1}(j), \square^k\big)$$

where $\square^k$ denotes the k-dimensional cube $[0,1]^k$, and Rect denote a rectilinear embedding, which means an injective map $\square^k \times f^{-1}(j) \to \square^k$ that is "linear" in each square (i.e. mapping a square to a parallelogram). This is the $\mathbb{E}_k$ operad we have mentioned in the introduction, and is of fundamental importance in homotopy theory. You may believe this or not, but the algebras for this operad in suitably nice ∞-categories (for example presentable ones) endowed with the Cartesian monoidal structure are precisely the groups commutative to the k-th level, which is another fancy term for k-th loopspaces.

For sharp readers who noticed that we haven't properly defined an ∞-category in terms of a quasi-category, we can simply take a so called nerve of a topological category and obtain a quasi-category.

We now turn to introduce an exotic (at least so when one first see it) example of ∞-operad, which serves as a natural generalization to the above $\mathbb{E}_k^{\otimes}$ operad.

Example 2.2.11. Let M be a topological manifold of dimension k. We define $\mathcal{C}_M$ to be the ∞-category with objects $\mathbb{R}^k$, M and mapping spaces given by

$$\begin{gathered}\operatorname{Hom}\big(\mathbb{R}^k, \mathbb{R}^k\big) = \operatorname{Emb}\big(\mathbb{R}^k, \mathbb{R}^k\big),\\ \operatorname{Hom}\big(\mathbb{R}^k, M\big) = \operatorname{Emb}\big(\mathbb{R}^k, M\big),\\ \operatorname{Hom}(M, M) = \operatorname{id}_M,\\ \operatorname{Hom}(M, \mathbb{R}^n) = \emptyset\end{gathered}$$

We let $\operatorname{Top}(k)$ denote the topological homeomorphisms $\mathbb{R}^k \to \mathbb{R}^k$. By definition, we have an inclusion of categories $B\operatorname{Top}(k) \hookrightarrow \mathcal{C}_M$. Then, let B_M denote the ∞-groupoid

$$B_M = B\operatorname{Top}(k) \times_{\mathcal{C}(M)} \mathcal{C}(M)_{/M}$$

Note in particular we have $B_M \simeq M$ as animas. Then, note that we may upgrade the structure of $B\operatorname{Top}(k)$ to an operad as follows: the objects of $B\operatorname{Top}(k)$ are $\langle n\rangle$, and the morphisms of $B\operatorname{Top}(k)$over $\alpha : \langle m\rangle \to \langle n\rangle \in \mathbf{Fin}_*$ are given by

$$\prod_{1\leqslant i\leqslant n} \operatorname{Emb}\big(\mathbb{R}^k \times \alpha^{-1}(i), \mathbb{R}^k\big)$$

Finally we let $\mathbb{E}_M^{\otimes} = B\operatorname{Top}(k)^{\otimes} \times_{B\operatorname{Top}(k)^{\amalg}} B_M^{\amalg}$ (here $X^{\amalg}$ is the free coCartesian operad generated by the anima X viewed as an ∞-groupoid, or an ∞-category with every morphism invertible).

By definition, we may compute the color $\mathbb{E}_M = M$, which means that an algebra of $\mathbb{E}_M^{\otimes}$ in some ∞-category $\mathcal{C}$ would first require a data of a map $M \to \mathcal{C}$, where again M is regarded as an ∞-groupoid.

Example 2.2.12. We define $\mathbb{E}_n^{\otimes} = \mathbb{E}_{\mathbb{R}^n}^{\otimes}$. Here, $\mathbb{R}^n$ is contractible, so an $\mathbb{E}_n^{\otimes}$-algebra in $\mathcal{C}$ is simply an object $x \in \mathcal{C}$ with some structural maps $x^{\otimes i} \to x$. In fact, as promised in the introduction, we have the following recognition principle.

Theorem 2.2.23 (recognition principle). Let $\mathcal{C}$ be an ∞-topos, regarded as a symmetric monoidal ∞-category via its Cartesian monoidal structure. Then, we have $\mathbf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathcal{C}_{*,\geqslant n}$, where $\mathcal{C}_{*,\geqslant n}$ is the subcategory of $\mathcal{C}$ spanned by the pointed n-connective objects, and the functor $\mathcal{C}_{*,\geqslant n} \to \mathbf{Alg}_{\mathbb{E}_n}(\mathcal{C})$ is given by the n-fold loop space $\Omega^n : \mathcal{C}_{*,\geqslant n} \to \mathbf{Alg}_{\mathbb{E}_n}(\mathcal{C})$.

From now on, for simplicity, we will denote $\mathbf{Alg}$ by $\mathbf{Alg}_{\mathbb{E}_1}$ and $\mathbf{CAlg}$ by $\mathbf{Alg}_{\mathbb{E}_\infty}$, called the associative algebras and commutative algebras.

The description of general $\mathbb{E}_M^{\otimes}$-algebras is extremely difficult, since it relies on the tangential structure of the manifold. Hence, for manifolds with trivial tangent bundle, the discription would be much easier. For example, the $E_{S^1}^{\otimes}$-algebras can be identified with associative algebras equipped with an involution. We will not go deep into this.

Then we sketch the theory of dendroidal sets.

remark 2.2.13. Briefly, recall that an ∞-category (at least in our model, the quasi-categories) is essentially an simplicial set with some nice lifting properties ensuring the arrows in the simplicial set compose. This condition is not that important, since every simplicial set is homotopic equivalent (at least in a suitable sense) to a ∞-category, so we temporarily ignore this condition. There is a more heuristic way of regarding a simplicial set as an ∞-category. By definition, a simplicial set is a presheaf of sets over the category of simplices Δ. If we think $[n] \in \Delta$ as the poset

$$0 \to 1 \to \cdots \to n$$

By the Yoneda lemma, we may regard for a simplicial set X the value taken at $[n]$, $X([n])$ or X_n, as the ways of $[n]$ mapping in to X, or more categorically, as the ways of how n morphisms in X compose modulo higher homotopy. Through this process, we realize that a simplicial set remembers the homotopical data of an ∞-categories by recording the possible ways of how n morphisms compose.

On the other hand, in classical category theory, we may change our perspective on the (colored) operads, and instead identifying them as the multicategories. Informally, a multicategory is a collection of objects, and for each set of objects $\{X_1, \cdots, X_n\}$ and an object Y, a set of "multilinear" maps $\mathrm{Hom}(\{X_1, \cdots, X_n\}, Y)$, which, translating to the operadic language, is precisely our previously defined $P(X_1, \cdots, X_n; Y)$ where X_i and Y are colors of the operad. So, according to the above intuition for remebering homotopical data of composing morphisms, we should remember all possible ways these multimorphisms compose. This is where the theory of dendroidal sets set in. In a multicategory, we may simply regard $\mathrm{Hom}(\{X_1, \cdots, X_n\}, Y)$ as the way of "mapping" a corolla tree C_n into it, and so, all the ways multimorphisms compose would simply be recorded by way of "mapping" a general tree T into a multicategory. This motivates completely the following definition of dendroidal sets.

Definition 2.2.24. We define the category of dendroidal sets to be the category of presheaves $\mathrm{pSh}(\Omega)$, where Ω is the category of trees defined in the previous section. We no longer assume T to be open and reduced now.

remark 2.2.14. The elementary morphisms introduced last section are precisely the analoges of the simplicial faces and degeneracies.

Definition 2.2.25. For a tree T and an elmentary face $\partial_x T$, we define the horn Λ_T^x to be the subdendroidal set of T obtained by the union of all the faces except x. This is the analogue of the simplicial horn $\Lambda_i^n \subset \Delta^n$.

Definition 2.2.26 ([HM22], Definition 6.1). We define a dendroidal set X to be an ∞-operad if it satisfies the lifting conditions for all inner horns $\Lambda_T^x \subset T$ where $\partial_x T$ is an inner face.

remark 2.2.15. Recall in the first section that we defined an operad to be a presheaf over certain trees subject to a certain grafting condition. Here, in the case of ∞-operads, this condition is equivalent to the above inner lifting condition due to that the monoidal structure is the Cartesian one, so we may take advantage of this and translate the condition to the lifting condition.

We may relate this definition to the previous definition of Lurie via the following theorem:

Theorem 2.2.27 ([HM24]). Recall that we have a fully faithful functor $F : \Omega \hookrightarrow \mathbf{Op}$. Then, the following dendroidal nerve functor is an equivalence of categories from the ∞-operads in the sense of Lurie to the dendroidal ∞-operads.

$$N_d(-) = \mathrm{Hom}_{\mathbf{Op}_{(\infty,1)}}(F(-), -) \in \mathrm{pSh}(\Omega)$$

In fact, there are a lot more different (but equivalent) models for the category of ∞-operads (for example, there is a suitable generalization of the category of symmetric sequences in ∞-categories), but for simplicity, we will just present the two models above.

remark 2.2.16. By the grafting condition (or the above equivalent lifting condition), notice that the ∞-operads again is equivalent to a subcategory of symmetric sequences. We will explain more in the next section.

2.3. Goodwillie calculus

The final categorical tool we will use in the construction of the $\mathbb{L}_\infty$-operad is the Goodwillie calculus. In some sense, Goodwillie calculus is the free functor from the category of functors to the category of analytic functors.

Definition 2.3.1. Recall that $[n]$ is the set of n elements. We let $\mathcal{P}([n])$ denote the set of subsets of $[n]$, which is a poset with respect to the inclusion of subsets. This poset looks like an n-dimensional cube. We define an n-cube in a category $\mathcal{C}$ to be a functor $\mathcal{P} \to \mathcal{C}$.

Construction 2.3.2. Let $X : \mathcal{P}([n]) \to \mathcal{C}$ be an n-cube in $\mathcal{C}$. We define its total fiber

$$\operatorname{tfib} X = \operatorname{fib}\left(X(\emptyset) \to \lim_{\substack{S \in \mathcal{P}([n]) \\ S \neq \emptyset}} X(S) \right)$$

Dually, we define its total cofiber

$$\operatorname{tcofib} X = \operatorname{cofib}\left(\operatorname*{colim}_{\substack{S \in \mathcal{P}([n]) \\ S \neq [n]}} X(S) \to X([n]) \right)$$

Proposition 2.3.3. Let $X : \mathcal{P}([n]) \to \mathcal{C}$ be a cube. We let $Y : \mathcal{P}([n-1]) \to \mathcal{C}$ denote the cube obtained by $Y(T) = \operatorname{fib}(X(T) \to X(T \cup n))$. Then, we have $\operatorname{tfib} X = \operatorname{tfib} Y$.

Dually, let $Y : \mathcal{P}([n-1]) \to \mathcal{C}$ denote the cube obtained by $Y(T) = \mathrm{cofib}(X(T) \to X(T \cup n))$. Then, we have $\mathrm{tcofib}\, X = \mathrm{tcofib}\, Y$

Corollary 2.3.4. Let $S \in \mathcal{P}([n])$ be a subset, and let $T = [n] - S$. Then for any n-cube $X : \mathcal{P}([n]) \to \mathcal{C}$, let Y be the S-cube obtained by for $S' \subset S$,

$$Y(S') = \mathrm{tfib}_{T' \subset T}\, X(S' \cup T')$$

we have $\mathrm{tfib}\, X = \mathrm{tfib}\, Y$. The dual statement holds for total cofibers.

Definition 2.3.5. We say a cube of spaces $X : \mathcal{P}([n]) \to \mathbf{Ani}$ to be Cartesian if its total fiber is the point. We say it is k-Cartesian if the total fiber is k-connected.

We say a cube of spaces to be coCartesian if its total cofiber is the point. We say it is k-coCartesian if the total cofiber is k-connected.

We say a cube of spaces to be strongly coCartesian if all its 2-faces are coCartesian squares.

Definition 2.3.6. For a functor $F : \mathbf{Ani}_* \to \mathbf{Ani}_*$, we say F satisfies $E_n(c, \kappa)$ if for any strongly coCartesian $(n+1)$-cube X with $X(\emptyset) \to X(i)$ κ-connective, we have $F(X)$ a c-Cartesian cube.

We say F is n-excisive if it satisfies $E_n(c, \kappa)$ for all c and κ.

Corollary 2.3.7. By definition, every n-excisive functor is $(n+1)$-excisive.

Definition 2.3.8. We say a functor F is ρ-analytic if there is some q such that F satisfies the property $E_n(n\rho - q, \rho + 1)$ for all n.

Example 2.3.1. By the Blakers-Messay theorem, we have the identity functor 1-analytic.

Theorem 2.3.9 ([Goo03]). Let F be a ρ-analytic functor. Then, there are n-excisive functors $P_n F$ equipped with comparison maps $F \to P_n F$. This is the couniversal n-excisive functor equipped with a natural transformation from F.

Moreover, we have the following convergence result: If a space X is $(\rho+1)$-connected, then the connectivity of the map $F(X) \to P_n F(X)$ converges to ∞ when n goes to ∞. That is, the sequence of functors

$$P_0 F \leftarrow P_1 F \leftarrow \cdots$$

converges for sufficiently connective spaces.

We make some more definitions that will help in our later computations.

Definition 2.3.10. We say a functor F is n-homogeneous if it is n-excisive and $P_{n-1}F = *$.

Theorem 2.3.11 ([Goo03]). All n-homogeneous functors are of the form

$$\Omega^\infty\Big(C \otimes (\Sigma^\infty(-))^{\otimes n}\Big)_{h\Sigma_n}$$

where C is a spectrum with Σ_n-action, and $(-)_{h\Sigma_n}$ is the homotopy quotient functor.

Definition 2.3.12. Let F be a functor either of spaces or spectra. We let

$$D_nF = \mathrm{fib}(P_nF \to P_{n-1}F)$$

We define the n-th derivative of F, ∂_nF, to be the spectrum C with Σ_n-action corresponding to D_nF as in the above theorem.

3. Koszul Duality

As we mentioned in the introduction, Koszul duality first appeared in Quillen's paper on rational homotopy theory ([Qui68]). In this paper, the main theorem is the following:

Theorem 3.1 (Quillen). The category $\mathbf{Ani}_{\mathbb{Q},\geqslant 2}$ of simply connected rational homotopy types of spaces is equivalent to both of the following:

- The category of differential graded Lie algebras, given by sending a space to its rational homotopy groups, equipped with the Whitehead product as the Lie bracket.
- The category of 2-reduced coalgebras, given by sending a space to its rational coefficient homology, with the coalgebra structure induced by the diagonal map $X \hookrightarrow X \times X$.

This theorem hints a certain duality relation between the Lie algebras and coalgebras, which is now known as Koszul duality. This duality was soon enhanced to the Koszul duality between the Lie operad and the commutative cooperad (a cooperad is simply an operad in the opposite category). In this chapter, we will introduce the Koszul duality of ∞-operads, and define the $\mathbb{L}_\infty$-operad by it. Moreover, we will give an alternative definition of the $\mathbb{L}_\infty$-operad via Goodwillie calculus.

We give a tiny remark here that the Koszul duality for operads only holds for reduced operads.

3.1. Quadratic duality

We begin the chapter with the most ancient version of Koszul duality, introduced by Kapranov and Ginzburg in [GK94]. In short, some operads are generated by binary operations modulo trinary axioms. In this case, the duality of vector spaces gives the desired Koszul duality. In this section, we work over the category $\mathbf{Vect}_k$.

Definition 3.1.1. Let K be an assoiciative k-algebra. We define a K-collection to be a collection $E = \{E(n)\}_{n\geqslant 2}$ of k-vector spaces with the following additional data and compatibilities:

(1) A left Σ_n-action on $E(n)$.
(2) A $(K, K^{\otimes n})$-bimodule structure on $E(n)$.
(3) For $s \in \Sigma_n$, s commutes with the left K-action and permutes the right $K^{\otimes n}$-action.

Construction 3.1.2. For any K-collection $\{E(n)\}$ we may define a corresponding free operad E defined via the following process. Firstly, we define for any tree T the k-vector space

$$E(T) = \bigotimes_{v\in T} E(I(v))$$

and finally define

$$E(n) = \bigoplus_{T} E(T)$$

where the T in the above sum runs through trees with n leaves.

This gives the definition of a symmetric sequence in $\mathbf{Vect}_k$. Moreover, it is a monoid by definition. Thus, this gives the construction of an operad.

remark 3.1.1. This definition may seem contradictory to the previous theorem relating the operads as presheaves and operads as symmetric sequences. However, recall that the grafting morphisms are all equivalences, so we have for an operad P the object $P(T)$ depend only on the number of leaves of T.

Definition 3.1.3. Let P be an operad (regarded as a symmetric sequence). We define its ideal to be a sub-symmetric sequence $I(n) \subset P(n)$ such that for $\lambda \in P(m)$ and $\lambda_i \in P(n_i)$ where $1 \leqslant i \leqslant m$, if any one of λ and λ_i lie in I, then the composition $\lambda(\lambda_1, \cdots, \lambda_m)$ lies in I.

Corollary 3.1.4. For P an operad and $I \subset P$ an ideal, the termwise quotient $P/I(n) = P(n)/I(n)$ admits an operad structure.

Construction 3.1.5. For a sub-symmetric sequence $R \subset P$, we may define the ideal (R) generated by R given by the minimal ideal of P containing R. One may also define it as the sub-symmetric sequence of R generated by multiplying with elements in I, like the case of rings.

Construction 3.1.6. We now construct a special type of operads.

Let K be a semisimple k-algebra. Let E be a $(K, K^{\otimes 2})$-bimodule with an Σ_2-action σ satisfying the axioms that an object $E(2)$ in a K-collection would satisfy. Then, let $\{E(n)\}$ denote the K-collection with $E(2) = E$ and $E(n) = 0$ for $n > 2$. Then let $F(E)$ denote the free operad generated by this K-collection. Then, by definition we have $F(E)(3) = \mathrm{ind}_{\Sigma_2}^{\Sigma_3}(E \otimes_K E)$. Then, let $R \subset F(E)(3)$ be a Σ_3-stable $(K, K^{\otimes 3})$-submodule. Finally, we construct $P(K, E, R) = P(E)/(R)$.

Definition 3.1.7. We define a quadratic operad to be an operad arising from the above construction.

Construction 3.1.8. For a quadratic operad $P = P(K, E, R)$, we define its Koszul dual $K(P)$ to be the quadratic operad $P(K^{\mathrm{op}}, E^\vee, R^\perp)$, where $E^\vee$ is the dual module $\mathrm{Hom}_K(E, K)$ regarded as a right K-module, so a left K^{op}-module. Finally, note that

$F(E^\vee)(3) = F(E)(3)^\vee$, so the notation $R^\perp$ simply denote the orthogonal complement in the dual space. Obviously for K finite dimensional and V finite dimensional, we have $K^2 = \mathrm{id}$.

Example 3.1.2. The operads **Assoc**, **Comm**, and **Lie** are actually quadratic operads. Notice that if we take $K = k$, and fix a set of basis of E, say $E = \bigoplus ke_i$. We regard these e_i as binary operations. So, when we take quotients of the set of the trinary operations $F(E)(3)/R$, we regard this as identifying some compositions of binary operations. We give an example on giving the quadratic structures of **Assoc**, **Comm**, and **Lie**.

We first consider **Assoc**. We show $\mathbf{Assoc} = P(k, k^2, R)$, where $k^2 \circlearrowleft \Sigma_2$ by permutation. We may regard $k^2 = \mathrm{Span}(x_1x_2, x_2x_1)$. Under this, we have $F(k^2)(3) = k^{12}$, which is the space spanned by the operations $x_{s(1)}\big(x_{s(2)}x_{s(3)}\big)$ and $\big(x_{s(1)}x_{s(2)}\big)x_{s(3)}$ for $s \in \Sigma_3$. Then, we let $R \subset k^{12}$ be spanned by the elements $x_i\big(x_jx_k\big) - \big(x_ix_j\big)x_k$. By definition, this is **Assoc**.

Then, we consider the operads **Comm** and **Lie**.

Recall that Σ_3 has 3 irreducible representations 1, **sgn**, and **std**. We show that $\mathbf{Comm} = P(k, k, \mathbf{std})$. Firstly, the k in the second place is equipped with the trivial Σ_2-action, so $F(k)(3) = 1 \oplus \mathbf{std}$. This is the 3-dimensional Σ_3-representation given by permutation, which we regard as the space generated by the three operations $(x_1x_2)x_3$, $(x_2x_3)x_1$, and $(x_3x_1)x_2$. Then by the commutataivity axioms, we should obviously have the difference of any two zero, so $R = \mathbf{std}$ the space of vectors $\sum x_i = 0$.

Then, we show that $\mathbf{Lie} = P(k, \mathbf{sgn}, \mathbf{std})$, where **sgn** is the 1-dimensional Σ_2-representation given by sign. We have $F(\mathbf{sgn})(3) = \mathbf{sgn} \oplus \mathbf{std}$, regarded as the space spanned by the operations $[[x_1, x_2], x_3], [[x_2, x_3], x_1], [[x_3, x_1], x_2]$. We notice that **std** is the subspace generated by the Jacobi identities.

Corollary 3.1.9. We have $K(\mathbf{Assoc}) = \mathbf{Assoc}$, $K(\mathbf{Comm}) = \mathbf{Lie}$, $K(\mathbf{Lie}) = \mathbf{Comm}$.

Proof: This is directly seen by the above presentation. □

Over the quadratic operads, there is a special type of algebras called quadratic algebras.

Construction 3.1.10. Let P be an operad regarded as a symmetric sequence. Let $K = P(1)$, which is an algebra under the composition morphism, and $P(n)$ admit natural K-module structures by the composition morphisms. Note that there is a forgetful functor $\mathbf{Alg}_P \to \mathbf{Mod}_K$ given by taking the underlying vector space, equipped with the natural K-module structure.

The above forgetful functor has a left adjoint, which is the free algebra functor $F_P : \mathbf{Mod}_K \to \mathbf{Alg}_P$, given by

$$V \mapsto \bigoplus_{n \geqslant 1} (P(n) \otimes_{K^{\otimes n}} V^n)_{\Sigma_n}$$

Definition 3.1.11. Let A be a P-algebra. We define an ideal in A to be a subspace $I \subset A$ such that for any $\mu \in P(n)$, and $a_1, \cdots, a_{n-1} \in A$, $i \in I$, we have $\mu(a_1, \cdots, a_{n-1}, i) \in I$, where this element comes from the P-algebra structure morphism

$$P(n) \otimes A^{\otimes n} \to \mathrm{Hom}(A^{\otimes n}, A) \otimes A^{\otimes n} \to A$$

We have by definition A/I still a P-algebra.

Construction 3.1.12. We now construct a special type of algebras. Again, let P be a quadratic operad $P(K, E, R)$. Take any K-module V, we have $F_P(V)$ the free P-algebra generated by V. Then, we consider the second piece of this free algebra, namely $(E \otimes_{K^{\otimes 2}} V^{\otimes 2})_{\Sigma_2}$. This is a K-bimodule, where the right K-action is given by the diagonal action. Then, for $S \subset (E \otimes_{K^{\otimes 2}} V^{\otimes 2})_{\Sigma_2}$ be a K-sub-bimodule, we let $A(V, S) = F_P(V)/(S)$, where (S) again denote the ideal generated by S.

Definition 3.1.13. We define a quadratic algebra over a quadratic operad P to be an algebra of the form $A(V, S)$.

Example 3.1.3. Again, when $K = k$, the quadratic algebras are instances of $P(K, E, R)$-algebras freely generated by some elements under some relations in S. For example, consider any finitely presented ring $A = k[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$, where each f_i is of degree 2. Then, we let $P = \mathbf{Comm}$, and take $V = \mathrm{Span}(x_1, \cdots, x_n)$, so we have

$$F_P(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}_{\Sigma_n} = \bigoplus_{n \geqslant 0} \mathrm{Sym}^n V = \mathrm{Sym}^* V = k[x_1, \cdots, x_n]$$

Then, take $S \subset \mathrm{Sym}^2 V$ to be generated by f_i. Thus we have $A(V, S) = k[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$.

Construction 3.1.14. For a quadratic operad P and a quadratic algebra $A = A(V, S)$, we define its Koszul dual $K(A)$ to be the $K(P)$-algebra $A(V^\vee, S^\perp)$.

Obviously, for finite dimensional V, we have $K^2 = \mathrm{id}$.

3.2. The bar and cobar construction

Now, we generalize the above definition to more general operads and algebras. In this section, we work with operads and algebras in general presentably symmetric monoidal stable ∞-categories. For readers who are not familiar with this, one may simply think the above as the ∞-categorical analogue of triangulated categories with limits and colimits.

Definition 3.2.1. Let $\mathcal{C}$ be any ∞-category. We define $\mathrm{SSeq}(\mathcal{C}) = \mathrm{Fun}(\mathbf{Fin}^{\simeq}, \mathcal{C})$.

Construction 3.2.2. We now upgrade the composition product to an monoidal structure on $\mathrm{SSeq}(\mathcal{C})$, for $\mathcal{C}$ a presentably symmetric monoidal stable ∞-category (recall previously it is only a monoidal structure in the 1-categorical sense, which has no higher coherences required).

Firstly, recall that $\mathcal{C}$ is the universal presentable $\mathcal{C}$-linear ∞-category, so any such $\mathcal{D}$ has a unique exact presentable map $\mathcal{C} \to \mathcal{D}$. Then, if we equip $\mathrm{SSeq}(\mathcal{C})$ with the Day convolution product, this category becomes the universal presentably symmetric monoidal $\mathcal{C}$-linear ∞-category over one object, that is, for any presentably symmetric monoidal $\mathcal{C}$-linear ∞-category $\mathcal{D}$, we have an equivalence of categories

$$\mathrm{Fun}^{\otimes,L}(\mathrm{SSeq}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

In particular, if we substitute $\mathcal{D} = \mathrm{SSeq}(\mathcal{C})$, we have an equivalence

$$\mathrm{Fun}^{\otimes,L}(\mathrm{SSeq}(\mathcal{C}), \mathrm{SSeq}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{SSeq}(\mathcal{C})$$

Then, the composition on LHS is naturally a monoidal structure. Passing this monoidal structure to RHS gives the monoidal structure of the composition product. In particular, the tensor unit is given by the unit symmetric sequence I with $I(1) = 1_{\mathcal{C}}$ and 0 otherwise.

Construction 3.2.3. There is an inclusion functor $\mathrm{SSeq}(\mathcal{C}) \hookrightarrow \mathrm{End}(\mathcal{C})$ given by sending a symmetric sequence S to the functor

$$\mathrm{Sym}_S : X \mapsto \bigoplus_{n \geqslant 1} \left(S(n) \otimes X^{\otimes n}\right)_{h\Sigma_n}$$

The composition product is the unique monoidal structure on $\mathrm{SSeq}(\mathcal{C})$ such that this inclusion is monoidal. Namely, we have $\mathrm{Sym}_{T \circ S} = \mathrm{Sym}_T \circ \mathrm{Sym}_S$.

Definition 3.2.4. For an operad P in $\mathcal{C}$, we define the category of P-algebras $\mathbf{Alg}_{P(\mathcal{C})}$ to be the category $\mathbf{Alg}_{\mathrm{Sym}_P}(\mathcal{C})$, where since the inclusion $\mathrm{SSeq}(\mathcal{C}) \hookrightarrow \mathrm{End}(\mathcal{C})$ is monoidal, we have for P an operad Sym_P a monad.

Definition 3.2.5. Recall we define $\mathbf{Alg}(\mathcal{C})$ to be the category of $\mathbb{E}_1$-algebras in $\mathcal{C}$. We define $\mathbf{coAlg}(\mathcal{C})$ to be the category of $\mathbb{E}_1$-coalgebras in $\mathcal{C}$, that is, $\mathbf{Alg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Unwinding definitions, an $\mathbb{E}_1$-coalgebra in $\mathcal{C}$ is equivalently an object C equipped with a counit map $\varepsilon : C \to 1_{\mathcal{C}}$ and a homotopy associative comultiplication $\Delta : C \to C \otimes C$.

Definition 3.2.6. Unwinding the definition of the dendroidal ∞-operad, the category of ∞-operads in $\mathcal{C}$ is identified with the category of algebras in $\mathrm{SSeq}(\mathcal{C})$ with the composition product. Namely, we define

$$\mathbf{Op}(\mathcal{C}) \coloneqq \mathbf{Alg}(\mathrm{SSeq}(\mathcal{C}))$$

For the complete notion of Koszul duality, we need the dual construction for operads.

Definition 3.2.7. We define a cooperad in $\mathcal{C}$ to be an operad in $\mathcal{C}^{\mathrm{op}}$, in which $\mathcal{C}^{\mathrm{op}}$ is equipped with the opposite monoidal structure. Using the definition $\mathbf{Op}(\mathcal{C}) = \mathbf{Alg}(\mathrm{SSeq}(\mathcal{C}))$, we have

$$\mathbf{coOp}(\mathcal{C}) = \mathbf{coAlg}(\mathrm{SSeq}(\mathcal{C}))$$

Definition 3.2.8. We define an operad or cooperad to be reduced if its underlying symmetric sequence S has $S([1]) = 1_{\mathcal{C}}$.

Now we give some common operations on algebras.

Construction 3.2.9. Let $f : P_1 \to P_2$ be a morphism of operads. Then, it induces $f : \mathrm{Sym}_{P_1} \to \mathrm{Sym}_{P_2}$ a morphism of monads. Recall that for a morphism of monads $f : T_1 \to T_2$, there is a pullback of algebras $f^* : \mathbf{Alg}_{T_2} \to \mathbf{Alg}_{T_1}$ by sending the morphism $T_2 B \to B$ to $T_1 B \to T_2 B \to B$, with the obvious algebra structure. This functor admits a left adjoint $f_! \dashv f^*$, where

$$f_! A = \operatorname*{colim}_{\Delta^{\mathrm{op}}} T_2 T_1^n A$$

the boundary maps are given by either $T_1^2 \to T_1$, $T_1 A \to A$, or $T_2 T_1 \to T_2 T_2 \to T_2$, and the degeneracies the unit maps of monads. $f_! A$ is a T_2-algebra since it is a colimit of free-T_2-algebras, and is the left adjoint since

$$\begin{aligned}
&\mathrm{Hom}_{T_2}(f_! A, B) \\
&= \mathrm{Hom}_{T_2}\left(\operatorname*{colim}_{\Delta^{\mathrm{op}}} T_2 T_1^n A, B\right) \\
&= \lim_{\Delta} \mathrm{Hom}_{T_2}(T_2 T_1^n A, B) \\
&= \lim_{\Delta} \mathrm{Hom}(T_1^n A, B) \\
&= \lim_{\Delta} \mathrm{Hom}_{T_1}\left(T_1^{n+1} A, B\right) \\
&= \mathrm{Hom}_{T_1}\left(\operatorname*{colim}_{\Delta^{\mathrm{op}}} T_1^{n+1} A, B\right) \\
&= \mathrm{Hom}_{T_1}(A, B)
\end{aligned}$$

The above construction, applied to the case of operads, gives an adjunction

$$f_! : \mathbf{Alg}_{P_1}(\mathcal{C}) \rightleftarrows \mathbf{Alg}_{P_2}(\mathcal{C}) : f^*$$

for any map $f : P_1 \to P_2$.

Dually, for a map of comonads $f : T_1 \to T_2$, there is a natural pushforward functor and its right adjoint given by totalization of the likewise diagram

$$f_* : \mathbf{coAlg}_{T_1}(\mathcal{C}) \rightleftarrows \mathbf{coAlg}_{T_2}(\mathcal{C}) : f^!$$

Notice that all our operads constucted before admit an augmentation, for example, there is a natural map of symmetric sequences $\mathbf{Assoc} \to I$. It turns out that it is the augmented versions of these operads that we are caring about, which is shown in the following constructions.

Construction 3.2.10. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Let $A \in \mathbf{Alg}(\mathcal{C})$ be an associative algebra equipped with an augmentation $A \to 1_{\mathcal{C}}$. We define its bar construction to be $\mathrm{Bar}(A) = 1_{\mathcal{C}} \otimes_A 1_{\mathcal{C}}$. Here, the relative tensor is the geometric realization of the simplicial object

$$1_{\mathcal{C}} \otimes_A 1_{\mathcal{C}} = \operatorname*{colim}_{\Delta^{\mathrm{op}}} \left(\cdots \xrightarrow{\cdots} A \otimes A \,\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\, A \,\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\, 1_{\mathcal{C}} \right)$$

This object has the natural structure of an augmented coalgebra.

The coalgebra structure is given by the following:

$$1_{\mathcal{C}} \otimes_A 1_{\mathcal{C}} \to 1_{\mathcal{C}} \otimes_A A \otimes_A 1_{\mathcal{C}} \to 1_{\mathcal{C}} \otimes_A 1_{\mathcal{C}} \otimes_A 1_{\mathcal{C}} \to (1_{\mathcal{C}} \otimes_A 1_{\mathcal{C}}) \otimes (1_{\mathcal{C}} \otimes_A 1_{\mathcal{C}})$$

where the middle map is induced by the augmentation of A.

The augmentation is the natural map

$$1_{\mathcal{C}} = 1_{\mathcal{C}} \otimes 1_{\mathcal{C}} \to 1_{\mathcal{C}} \otimes_A 1_{\mathcal{C}}$$

Dually, for a augmented coalgebra C, we may consider the cobar construction $\mathrm{coBar}(C)$ to be the totalization of the cosimplicial object

$$\mathrm{coBar}(C) = \lim_{\Delta} \left(1_{\mathcal{C}} \,\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\, C \,\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\, C \otimes C \xrightarrow{\cdots} \cdots \right)$$

The two functors admit a mild generalization: for an algebra A, a left A-module M, and a right A-module N, we denote $\mathrm{Bar}(M, A, N)$ by the geometric realization of the simplicial object as in $\mathrm{Bar}(A)$, but tensoring with a M on the right and N on the left. Dually, for a coalgebra C, a left C-comodule M, and a right C-comodule N, $\mathrm{coBar}(M, C, N)$ the likewise totalization.

We denote by $\mathrm{Bar}_{n(M,A,N)}$ the n-th object in the above diagram, and $\mathrm{Bar}_{\leqslant n}(M,A,N)$ the colimit taken to the diagram restricted to the first n-elements. The identical definitions are made for coBar.

In the above sense, the augmentation of the (co)algebras may be regarded as giving a module structure on $1_{\mathcal{C}}$.

Theorem 3.2.11 ([Lur17], Theorem 5.2.2.17). We have an equivalence of mapping spaces

$$\mathrm{Hom}_{\mathbf{Alg}(\mathcal{C})_{/1_{\mathcal{C}}}}(A, \mathrm{coBar}(C)) \simeq \mathrm{Hom}_{\mathbf{coAlg}(\mathcal{C})_{1_{\mathcal{C}}/}}(\mathrm{Bar}(A), C)$$

Moreover, both space admit the following moduli description: both spaces are the moduli space of lifting the pair (A, C).

In our setting, we take $\mathcal{C} = \mathrm{SSeq}(\mathcal{C})$, so Bar and coBar induces an adjunction between augmented operads and augmented cooperads. In fact, in our setting, a better conclusion holds: Bar and coBar are actually equivalence of categories inverse to each other, after restricting to a nicer subcategory.

Theorem 3.2.12 ([Gij24], Theorem 3.4). The above Bar-coBar adjunction restricts to an equivalence on the categories of reduced operads and reduced cooperads.

Proof: We prove for a reduced operad P we have the counit map $P \to \mathrm{coBar}(\mathrm{Bar}(P))$ is an equivalence. The counit $\mathrm{Bar}(\mathrm{coBar}(Q)) \to Q$ is likewise an equivalence.

Let $\tau_n P$ denote the truncation of P to stage n, which is the symmetric sequence with $\tau_n P([k]) = P([k])$ for $k \leqslant n$, and 0 otherwise. Then, we obviously have $\tau_n P$ a right P-module. We claim that we have an equivalence

$$\tau_n P \xrightarrow{\simeq} \mathrm{coBar}(\mathrm{Bar}(\tau_n P, P, I)), \mathrm{Bar}(P), I)$$

Firstly note that this equivalence implies the result since $\lim \tau_n P = P$, and

$$\lim(\mathrm{Bar}(\tau_n P, P, I)) = \mathrm{Bar}(\lim \tau_n P, P, I)) = \mathrm{Bar}(P, P, I) = I$$

Then, for this equivalence, we apply induction on n. For the base case $n = 1$, since P is reduced, we have $\tau_1 P = I$, the unit symmetric sequence. So, the equivalence is simply

$$I = \mathrm{coBar}(\mathrm{Bar}(P), \mathrm{Bar}(P), I) = \mathrm{coBar}(\mathrm{Bar}(I, P, I), \mathrm{Bar}(P), I) = \mathrm{coBar}(\mathrm{Bar}(\tau_1 P, P, I), \mathrm{Bar}(P), I)$$

Then, for the induction step, consider the fiber sequence of morphisms

$$\begin{tikzcd}
P(n) \arrow[r] \arrow[d] & \mathrm{coBar}(\mathrm{Bar}(P(n), P, I), \mathrm{Bar}(P), I) \arrow[d] \\
\tau_n P \arrow[r] \arrow[d] & \mathrm{coBar}(\mathrm{Bar}(\tau_n P, P, I), \mathrm{Bar}(P), I) \arrow[d] \\
\tau_{n-1} P \arrow[r] & \mathrm{coBar}(\mathrm{Bar}(\tau_{n-1} P, P, I), \mathrm{Bar}(P), I)
\end{tikzcd}$$

It suffices to prove the top row is an equivalence. By the presentability of the monoidal structure, we have $\mathrm{Bar}(P(n), P, I) = P(n) \circ \mathrm{Bar}(P)$. Then, we have

$$\mathrm{coBar}(P(n) \circ \mathrm{Bar}(P), \mathrm{Bar}(P), I) = P(n)$$

by explicitly writing down the contracting homotopy in the computation of the limit, analogous to the proof of $\mathrm{coBar}(C, C, I) = I$. □

The above construction gives the Koszul duality for operads, generalizing vastly the quadratic duality. However, the duality functor on algebras turns out to be much more subtle than the case of quadratic algebras.

Recall in the above proof, we used the symmetric sequences $\tau_n P$ to approximate P. We shall make more use of this construction in the following.

Definition 3.2.13. We define the category of n-truncated symmetric sequences to be

$$\mathrm{SSeq}_{\leqslant n}(\mathcal{C}) = \mathrm{Fun}(\mathbf{Fin}^{\simeq}_{\leqslant n}, \mathcal{C})$$

Which is monoidal equipped with the truncated composition product. We define the category of n-truncated operads to be $\mathbf{Op}_{\leqslant n}(\mathcal{C}) = \mathbf{Alg}(\mathrm{SSeq}_{\leqslant n}(\mathcal{C}))$. Dually, we define $\mathbf{coOp}_{\leqslant n}(\mathcal{C}) = \mathbf{coAlg}(\mathrm{SSeq}_{\leqslant n}(\mathcal{C}))$.

Construction 3.2.14. Obviously, one has a restriction functor $\rho_n : \mathrm{SSeq}(\mathcal{C}) \to \mathrm{SSeq}_{\leqslant n}(\mathcal{C})$. Clearly we have the direct sum decomposition $\mathrm{SSeq}(\mathcal{C}) = \mathrm{SSeq}_{\leqslant n}(\mathcal{C}) \oplus \mathrm{SSeq}_{>n}(\mathcal{C})$, so the extension by zero functor $\zeta_n : \mathrm{SSeq}_{\leqslant n}(\mathcal{C}) \to \mathrm{SSeq}(\mathcal{C})$ is both a left and a right adjoint. In particular, since ρ_n is a monoidal functor, ζ_n is both a lax (as a right adjoint) and an op-lax monoidal functor (as a left adjoint).

These functors induces corresponding functors on operads. By the adjoint functor theorem, the functors $\rho_n : \mathbf{Op}(\mathcal{C}) \to \mathbf{Op}_{\leqslant n}(\mathcal{C})$ and $\rho_n : \mathbf{coOp}(\mathcal{C}) \to \mathbf{coOp}_{\leqslant n}(\mathcal{C})$ admits both left and right adjoints. We now explicitly construct them. Firstly, since lax monoidal functors preserves algebra objects and colax monoidal functors preserves coalgebra objects, we have

$$\tau_n = \zeta_n|_{\mathbf{Op}_{\leqslant n}(\mathcal{C})} : \mathbf{Op}_{\leqslant n}(\mathcal{C}) \to \mathbf{Op}(\mathcal{C})$$

right adjoint to ρ_n, and

$$\tau^n = \zeta_n|_{\mathbf{coOp}_{\leqslant n}(\mathcal{C})} : \mathbf{coOp}_{\leqslant n}(\mathcal{C}) \to \mathbf{coOp}(\mathcal{C})$$

left adjoint to ρ_n. As in the previous proof, they are the extension by zero functor of operads and cooperads.

On the other hand, the left adjoint of $\rho_n : \mathbf{Op}(\mathcal{C}) \to \mathbf{Op}_{\leqslant n}(\mathcal{C})$ is a bit more mysterious. This functor is denoted by φ_n, and is the free operad functor generated by operations with arity lesser or equal to n. The same applies to $\rho_n \dashv \varphi^n$ for cooperads.

We abuse notations and abbreviate $\tau_n \rho_n X$ by $\tau_n X$. The rest functors all admit similar abbreviations.

To sum up, we have $\varphi_n \dashv \rho_n \dashv \tau_n$ for operads and truncated operads, and $\tau^n \dashv \rho_n \dashv \varphi^n$ for cooperads.

Definition 3.2.15. Let $F \in \mathrm{End}(\mathcal{C})$ be an endofunctor. We denote $\mathrm{alg}_F(\mathcal{C})$ the category of objects $c \in \mathcal{C}$ equipped with a map $F(c) \to c$.

Definition 3.2.16. Let S be a symmetric sequence in $\mathcal{C}$. We define

$$D_n^S(X) = (S(n) \otimes X^{\otimes n})_{h\Sigma_n}, D_S^n(X) = (S(n) \otimes X^{\otimes n})^{h\Sigma_n}$$

Proposition 3.2.17. We have a Cartesian square of ∞-categories

$$\begin{tikzcd} \mathbf{Alg}_{\varphi_n P}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{alg}_{D_n^P}(\mathcal{C}) \arrow[d] \\ \mathbf{Alg}_{\varphi_{n-1} P}(\mathcal{C}) \arrow[r] & \mathrm{alg}_{D_n^{\varphi_{n-1}P}}(\mathcal{C}) \end{tikzcd}$$

Moreover, taking the natural limit against the left maps gives $\mathbf{Alg}_P(\mathcal{C}) = \lim \mathbf{Alg}_{\varphi_n P}(\mathcal{C})$.

remark 3.2.1. We omit the proof of this proposition, but we state it here that this proposition is merely saying that an algebra of P is a tower of algebras A_n of $\varphi_n P$, such that the pullback of A_n along the natural inclusion $\varphi_{n-1} P \hookrightarrow \varphi_n P$, coincides with A_{n-1}. Moreover, the data of an $\varphi_n P$-algebra A_n restricting to a $\varphi_{n-1} P$-algebra A_{n-1} can be identified with a map

$$(P(n) \otimes X^{\otimes n})_{h\Sigma_n} \to X$$

compatible with the map

$$(\varphi_{n-1} P(n) \otimes X^{\otimes n})_{h\Sigma_n} \to X$$

(recall $\varphi_{n-1} P(n)$ is the subobject of $P(n)$ regarded as the subobject "spanned" by the operations with lower arity).

The coalgebras should be taken with more caution. Initially, we defined the coalgebras for $\mathbb{E}_1$ by the $\mathbb{E}_1$-algebras in the opposite categories. However, this definition does not work well generally, essentially because the opposite of presentable categories are not presentable, with the presentability of the monoidal structure especially broken. Hence, this definition does not work well for general cooperads. Here, we introduce the notion of divided power cooperads, which would appear on the dual side of the Koszul duality.

Definition 3.2.18. For a cooperad Q, we define $\widehat{\mathrm{Sym}}_Q$ to be the functor

$$\widehat{\mathrm{Sym}}_Q : X \mapsto \prod_{n \geqslant 1} (Q(n) \otimes X^{\otimes n})_{h\Sigma_n}$$

However, the functor $\widehat{\mathrm{Sym}}_{(-)}$ is only lax monoidal, so it sends operads to monads but not coalgebras to comonads, and the previous formalism would not apply to define a special class of coalgebras.

The source of the problem is that the products does not generally commutes with limits. To resolve this, we must pass to the pro-category.

Definition 3.2.19. Let $\mathcal{C}$ be an ∞-category. We define the category $\mathrm{Pro}(\mathcal{C})$ to be the category of all cofiltered diagrams in $\mathcal{C}$. There is a pair of adjoint functors $\mathcal{C} \rightleftarrows \mathrm{Pro}(\mathcal{C})$ given by the following: There is a functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ sending an object to the constant diagram with values in this object, and its right adjoint given by sending a diagram to its limit.

remark 3.2.2. Officially, $\mathrm{Pro}(\mathcal{C})$ is defined as the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathbf{Ani})^{\mathrm{op}}$ of the accessible preserving finite limits. The equivalence of definitions is given in Lurie's book [Lur08].

Theorem 3.2.20 ([Lur08]). The category $\mathrm{Pro}(\mathcal{C})$ admits the following universal property: for any category $\mathcal{D}$ with small cofiltered limits, we have

$$\mathrm{Fun}^{\mathrm{filt}}(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where the natural functor is obtained by pulling back along taking constant diagrams.

Hence, taking $\mathcal{D} = \mathrm{Pro}(\mathcal{C})$, we have

Corollary 3.2.21. We have an equivalence of categories

$$\mathrm{End}^{\mathrm{filt}}(\mathrm{Pro}(\mathcal{C})) = \mathrm{Fun}(\mathcal{C}, \mathrm{Pro}(\mathcal{C}))$$

Definition 3.2.22. For any functor $F : \mathcal{C} \to \mathcal{C}$, we define $pF : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C})$ to be the functor obtained by first composing F with the natural inclusion $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$, and then applying the above corollary to obtain $pF \in \mathrm{End}^{\mathrm{filt}}(\mathrm{Pro}(\mathcal{C}))$.

Note this functor p is symmetric monoidal, since the monoidal structure on both sides are given by composition.

Construction 3.2.23. Let $\mathrm{Sym}^{\leqslant n} : \mathrm{SSeq}(\mathcal{C}) \to \mathrm{End}(\mathcal{C})$ denote the functor

$$A \mapsto \bigoplus_{k \leqslant n} D_k^A$$

Then, we obtain a lax monoidal functor $p\,\mathrm{Sym}^{\leqslant n} : \mathrm{SSeq}(\mathcal{C}) \to \mathrm{End}^{\mathrm{filt}}(\mathrm{Pro}(\mathcal{C}))$. Finally, since the lax monoidalness of functors are preserved under limits,

$$\widehat{p\,\mathrm{Sym}} = \lim_n p\,\mathrm{Sym}^{\leqslant n}$$

is a lax monoidal functor.

Moreover, this functor is actually monoidal by category theory, essentially because $p\,\mathrm{Sym}^{\leqslant n}$ preserves cofiltered limits by definition.

Definition 3.2.24. By the above construction, for any operad Q, we have $\widehat{p\,\mathrm{Sym}}_Q$ a comonad, so we may define the $\widehat{p\,\mathrm{Sym}}_Q$-coalgebras in the category $\mathrm{Pro}(\mathcal{C})$.

We define the category of divided power coalgebras to be $\mathcal{C} \cap \mathbf{coAlg}_{\widehat{p\,\mathrm{Sym}}_Q}(\mathrm{Pro}(\mathcal{C}))$, or more rigorously the pullback

$$\begin{tikzcd} \mathbf{coAlg}_Q^{\mathrm{dp}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathbf{coAlg}_{\widehat{p\,\mathrm{Sym}}_Q}(\mathrm{Pro}(\mathcal{C})) \arrow[d] \\ \mathcal{C} \arrow[r] & \mathrm{Pro}(\mathcal{C}) \end{tikzcd}$$

Proposition 3.2.25. Dually, for a functor $F \in \mathrm{End}(\mathcal{C})$, we define $\mathrm{coalg}_F(\mathcal{C})$ to be the objects in $\mathcal{C}$ admitting a morphism $c \to F(c)$. Then, we have a pullback square

$$\begin{tikzcd} \mathbf{coAlg}_{\varphi^n Q}^{\mathrm{dp}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{coalg}_{D_n^Q}(\mathcal{C}) \arrow[d] \\ \mathbf{coAlg}_{\varphi^{n-1} Q}^{\mathrm{dp}}(\mathcal{C}) \arrow[r] & \mathrm{coalg}_{D_n^{\varphi^{n-1}Q}}(\mathcal{C}) \end{tikzcd}$$

Moreover, the natural map

$$\mathbf{coAlg}_Q^{\mathrm{dp}}(\mathcal{C}) = \lim_n \mathbf{coAlg}_Q^{\mathrm{dp}}(\mathcal{C})$$

This would be the coalgebras we will be considering.

Theorem 3.2.26 ([Gij24]). Let Q be a cooperad. Then, the category of ind-conilpotent (we havn't defined this) divided power coalgebras is equivalent to the Sym_Q-coalgebras (recall $\mathrm{Sym}_{(-)}$ is monoidal so in particular preserves coalgebra objects). Writing out, we have an equivalence

$$\mathbf{coAlg}_Q^{\mathrm{dp,nil}}(\mathcal{C}) \simeq \mathbf{coAlg}_{\mathrm{Sym}_Q}(\mathcal{C})$$

So for the coalgebras, this actually takes us back to the construction in the beginning.

We now construct the functors that would relate P-algebras to $\mathrm{Bar}(P)$-coalgebras and vice versa.

Construction 3.2.27. Let P be a reduced operad. Then, the projection $P \to \tau_1 P = I$ gives an augmentation of P. That is, we have morphisms $i : I \to P$ and $t : P \to I$ that composes to id_I. These morphisms give rise to the following adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "i_!"] & \mathbf{Alg}_P(\mathcal{C}) \arrow[l, shift left, "i^*"] \arrow[r, shift left, "t_!"] & \mathcal{C} \arrow[l, shift left, "t^*"] \end{tikzcd}$$

where since $ti = \mathrm{id}_I$, the horizontal arrows compose to $\mathrm{id}_{\mathcal{C}}$ (here in the diagram, we made the identification $\mathbf{Alg}_I(\mathcal{C}) \simeq \mathcal{C}$). Now, we rename the adjoint functors to provide more intuition, so we have the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\mathrm{free}_P"] & \mathbf{Alg}_P(\mathcal{C}) \arrow[l, shift left, "\mathrm{forget}_P"] \arrow[r, shift left, "\mathrm{cot}_P"] & \mathcal{C} \arrow[l, shift left, "\mathrm{triv}_P"] \end{tikzcd}$$

where free_P is the free P-algebra functor, forget_P the functor forgetting the P-algebra structure, and triv_P is the trivial P-algebra functor (by inducing the trivial multiplication on some object $c \in C$, seen by $\mathrm{forget}_P \mathrm{triv}_P = \mathrm{id}_{\mathcal{C}}$).

remark 3.2.3. The most mysterious functor is cot_P, the functor of taking cotangent fibers. To provide a first intuition, for $\mathcal{C} = \mathbf{Mod}_k$ the derived category over a base ring k, and P the $\mathbb{E}_\infty$-operad with A a commutative k-algebra, we have $\mathrm{cot}_{\mathbb{E}_\infty}(A) = \mathbb{L}_{A/k} \otimes_A k$ almost the cotangent complex.

Proposition 3.2.28. The comonad $\mathrm{cot}_P \mathrm{triv}_P$ obtained by the adjunction can be identified with the comonad arising from the cooperad $\mathrm{Sym}_{\mathrm{Bar}(P)}$. Thus, we have an equivalence of categories

$$\mathbf{coAlg}_{\mathrm{cot}_P \mathrm{triv}_P}(\mathcal{C}) \simeq \mathbf{coAlg}_{\mathrm{Bar}(P)}^{\mathrm{dp,nil}}(\mathcal{C})$$

Proof: Firstly, let X be a P-algebra. By definition, $\mathrm{Sym}_P = \mathrm{free}_P$. Then, by explicitly constructing the contracting homotopy, we have $\mathrm{Bar}(\mathrm{Sym}_P, \mathrm{Sym}_P, X) = X$ (here we slightly abuse notations and let this Bar construction be defined as changing the tensors in the previous Bar construction to composition and application of funcotrs). In this bar construction, we must identify Sym_P with $\mathrm{free}_P \mathrm{forget}_P$, to keep the categories compatible. Then, applying cot_P on both sides of the equation, we obtain

$$\mathrm{cot}_P X = \mathrm{Bar}(\mathrm{cot}_P \mathrm{Sym}_P, \mathrm{Sym}_P, X) = \mathrm{Bar}(\mathrm{id}, \mathrm{forget}_P \mathrm{free}_P, \mathrm{forget}_P X) = \mathrm{Bar}(\mathrm{id}, \mathrm{Sym}_P, \mathrm{forget}_P X)$$

Finally, substituting $\mathrm{triv}_P X$ into the above, we have

$$\mathrm{cot}_P \mathrm{triv}_P X = \mathrm{Bar}(\mathrm{id}, \mathrm{Sym}_P, X) = \mathrm{Sym}_{\mathrm{Bar}(P)}(X)$$

where the last equivalence is given by commuting the two different colimits, one over Δ^{op} and the other over $\mathbf{Fin}^{\simeq}$. $\square$

Construction 3.2.29. By general formalism of comonads, the functor $\mathrm{cot}_P : \mathbf{Alg}_{P(\mathcal{C})} \to \mathcal{C}$ factors through the category of $\mathrm{cot}_P \mathrm{triv}_P$-coalgebras, so we consider the diagram

$$\begin{tikzcd} & \mathbf{coAlg}^{\mathrm{dp,nil}}_{\mathrm{Bar}(P)}(\mathcal{C}) \arrow[d, "\mathrm{forget}^{\mathrm{dp,nil}}_{\mathrm{Bar}(P)}"] \\ \mathbf{Alg}_P(\mathcal{C}) \arrow[ur, "\mathrm{indec}^{\mathrm{nil}}_P"] \arrow[r, "\mathrm{cot}_P"'] & \mathcal{C} \end{tikzcd}$$

We may explicitly construct a right adjoint for $\mathrm{indec}^{\mathrm{nil}}_P$, given by

$$\mathrm{prim}^{\mathrm{nil}}_{\mathrm{Bar}(P)}(Y) = \mathrm{coBar}\Big(\mathrm{triv}_P, \mathrm{Sym}_{\mathrm{Bar}(P)}, \mathrm{forget}^{\mathrm{dp,nil}}_{\mathrm{Bar}(P)} Y\Big)$$

which is the right adjoint by similar proofs as above.

With some work on category theory, especially the formalism on filtered objects, we may lift the above adjunction to the following:

Proposition 3.2.30. There is an adjunction

$$\mathrm{indec}_P : \mathbf{Alg}_P(\mathcal{C}) \rightleftarrows \mathbf{coAlg}^{\mathrm{dp}}_{\mathrm{Bar}(P)}(\mathcal{C}) : \mathrm{prim}_{\mathrm{Bar}(P)}$$

To formulate the final result, we will take one more technical assumption on our category of (co)algebras. This is called the nilcompleteness conditions, which basically ensures the algebras behaves nicely under the approximation of (co)operads.

Definition 3.2.31. Recall we have morphisms $P \to \tau_n P$ and $\tau_n Q \to Q$ for operads and cooperads respectively. These maps induces adjuntions on the category of P-algebras or Q-coalgebras. We let $\mathrm{id} \to t_n$ denote the unit of the adjunction for $P \to \tau_n P$ and $t^n \to \mathrm{id}$ the counit of the adjunction for $\tau_n Q \to Q$.

We define an P-algebra X to be nilcomplete if $X \to \lim_n t_n X$ is an equivalence, and an Q-coalgebra Y to be nilcomplete if $\operatorname{colim}_n t^n Y \to Y$ is an equivalence. Moreover, we define $X \to \lim_n t_n X$ to be the nilcompletion of X and $\operatorname{colim} t^n Y \to Y$ the nilcompletion of Y. The two nilcompletion functors are idempotent, and projects the category of (co)algebras to the nilcomplete (co)algebras.

Theorem 3.2.32 ([Gij24]). Consider the adjunction

$$\mathrm{indec}_P : \mathbf{Alg}_P(\mathcal{C}) \rightleftarrows \mathbf{coAlg}^{\mathrm{dp}}_{\mathrm{Bar}(P)}(\mathcal{C}) : \mathrm{prim}_{\mathrm{Bar}(P)}$$

The unit of the adjunction is given by the nilcompletion of P-algebras, and the counit of the adjunction is given by the nilcompletion of divided power $\mathrm{Bar}(P)$-coalgebras.

remark 3.2.4. However, the functors $\mathrm{indec}^{\mathrm{nil}}_P$ and $\mathrm{prim}^{\mathrm{nil}}_{\mathrm{Bar}(P)}$ does not restricts to an equivalence on the nilcomplete (co)algebras, which basically states that for some cooperad Q there exists a non-nilpotent but nilcomplete coalgebra.

We omit the proof of this theorem, since we lack the discussion of graded and filtered algebras used in the lifting of $\mathrm{indec}^{\mathrm{nil}}_P$ to indec_P and its right adjoint.

3.3. The definition of Lie algebras

Definition 3.3.1. For an operad P, we define its operadic suspension ΣP to be the operad having the underlying symmetric sequence

$$\Sigma P(n) = \mathbb{S}^{\rho(n)} \otimes P(n)$$

where $\mathbb{S}^{\rho(n)}$ is the representation sphere of the $n-1$-dimensional standard Σ_n-representation. The operadic structure is given by the operadic structure of $P(n)$ and the compatibilities between $\mathbb{S}^{\rho(n)}$.

In particular, for P an operad concentrated at degree 0, ΣP is an operad concentrated at degree $d-1$.

Definition 3.3.2. For an operad P in $\mathcal{C}$ which is term-wise a dualizable object, we define its dual $P^\vee$ to be the cooperad with

$$P^\vee(n) = \mathrm{Map}(P(n), 1_{\mathcal{C}})$$

(here Map is the internal Hom of $\mathcal{C}$) and the cooperad structure given by the natural map

$$\mathrm{Hom}(c, e) \otimes \mathrm{Hom}(d, e) \xrightarrow{\simeq} \mathrm{Hom}(c \otimes d, e)$$

Theorem 3.3.3 ([Chi05]). We have an equivalence of ∞-cooperads in $\mathbf{Sp}$

$$\mathrm{Bar}(\mathbf{Comm}) = (\Sigma\,\mathbf{Lie})^\vee$$

Here $\mathbf{Lie}(n) = \mathbb{S}^{\otimes(n-1)!}$ equipped with the same Σ_n-action as the classical construction.

Corollary 3.3.4. By the previous section, we have $\mathrm{Bar}(\mathbf{Lie}) = (\Sigma\,\mathbf{Comm})^\vee$.

The computation of the homology of the Lie operad can thus be resolved by computing the Bar construction of the commutative operad, which has the easiest possible structure. However, there is another alternative construction of the shifted Lie algebra, given by the Goodwillie derivative of the identity. This gives another possible computation of the homology of the Lie operad.

Theorem 3.3.5 ([Lur17], Conjecture 6.3.0.7, actually proven in the book).
(1) We have $\partial_* \mathrm{id}_{\mathbf{Ani}}$ a symmetric sequence in spectra. Moreover, it is an operad.
(2) The Goodwillie derivative functor is a monoidal functor

$$\partial_* : \mathrm{Hom}(\mathbf{Ani}, \mathbf{Ani}) \to \mathbf{BiMod}_{\partial_* \mathrm{id}}(\mathrm{SSeq}(\mathbf{Sp}))$$

Theorem 3.3.6 ([Chi05]). The Goodwillie derivative of the identity is equivalent to the shifted Lie operad. Namely, we have an equivalence of operads $\partial_* \mathrm{id}_{\mathbf{Ani}} \to \Sigma\,\mathbf{Lie}$.

The above isomorphism gives a calculation for the homology, or say, spectral structure of $\partial_* \mathrm{id}_{\mathbf{Ani}}$. On the other hand, the calculation of $\partial_* \mathrm{id}_{\mathbf{Ani}}$ is an older problem studied by Joyal and other homotopists. We have the theorem of Johnson

Theorem 3.3.7 ([Bre95]). The homotopy type of $\partial_n \mathrm{id}_{\mathbf{Ani}}$ is given by the sum of $(n-1)!$ copies of the $(1-n)$-sphere spectrum.

Except for our most beloved operads, we present another important example of the computation of Koszul dual operads.

Theorem 3.3.8 ([CS22]). We have

$$\mathrm{Bar}(\mathbb{E}_n) = (\Sigma^n \mathbb{E}_n)^\vee$$

generalizing the computation on $\mathrm{Bar}(\mathbf{Assoc})$ since $\mathbf{Assoc} = \mathbb{E}_1$.

This fact will be used in the next chapter.

The algebra of **Lie** in other categories are of much importance, as we will also see in the next chapter.

Definition 3.3.9. A spectral Lie algebra is an algebra of **Lie** in the category **Sp**.

Let k be a characteristic 0 field. We define an $\mathbb{L}_\infty$-algebra over k to be an algebra of **Lie** in $\mathbf{Mod}_k$.

Example 3.3.1. A DGLA (differential graded Lie algebra) is a special type of $\mathbb{L}_\infty$-algebra. This is the type of $\mathbb{L}_\infty$-algebra most close to classical Lie algebras. Namely, a DGLA consists of the data of a chain complex L equipped with a bracket $[-,-] : L^{\otimes 2} \to L$. This bracket, except for the graded skew-symmetric constraint and the Jacobi identity, must satisfy another axiom: the derivation d should be a derivation of the bracket, namely

$$d[x_1, x_2] = [dx_1, x_2] - (-1)^{\deg x_1 \deg x_2}[dx_2, x_1]$$

remark 3.3.2. If we list out all the axioms of the DGLAs, we would arrive at the following three axioms on L (here we temporarily regard L just as a graded vector space, and d is a degree -1 map on L):
(1) $d(d(x_1)) = 0$
(2) $d[x_1, x_2] - \big([dx_1, x_2] - (-1)^{\deg x_1 \deg x_2}[dx_2, x_1]\big) = 0$
(3) $[[x_1, x_2], x_3] - (-1)^{\deg x_2 \deg x_3}[[x_1, x_3], x_2] + (-1)^{\deg x_1(\deg x_2 + \deg x_3)}[[x_2, x_3], x_1] = 0$

Now, one may notice that a DGLA is simply a $\mathbb{L}_\infty$-algebra with higher homotopy coherences identically 0. If we would restore these higher coherences (not setting them 0), we will obtain the classical definition of an $\mathbb{L}_\infty$-algebra.

Definition 3.3.10. Let $p, q \geqslant 0$. We define a (p,q)-shuffle to be a permutation $\sigma \in \Sigma_{p+q}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p), \text{and } \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$$

We define the set of (p,q)-shuffles to be $S(p,q) \subset \Sigma_{p+q}$.

Definition 3.3.11. Let V be a graded vector space. We define the Koszul sign conventions as the following:
(1) We let $s : V \otimes V \to V \otimes V$ given by the usual swap map $u \otimes v \mapsto (-1)^{\deg u \deg v} v \otimes u$.
(2) For any $\sigma \in \Sigma_n$, the swap map above induces an n-ary swap map

$$s_\sigma : V^{\otimes n} \to V^{\otimes n}$$

by decomposing σ in to products of 2-cycles.
(3) Let $\sigma \in \Sigma_n$. We define $\varepsilon(\sigma, v_1, \cdots, v_n) = \pm 1$ to be the value such that the identity

$$s_\sigma(v_1 \otimes \cdots \otimes v_n) = \varepsilon(\sigma, v_1, \cdots, v_n) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

holds.

(4) Finally, we define $\chi(\sigma, v_1, \cdots, v_n) = (-1)^\sigma \varepsilon(\sigma, v_1, \cdots, v_n)$, where $(-1)^\sigma$ denotes the number of inverses in σ.

Construction 3.3.12. Now we may formulate the equivalent definition for the $\mathbb{L}_\infty$-algebras. Let k be a characteristic 0 field and L be a graded k-vector space. We define an $\mathbb{L}_\infty$-algebra structure on L to be a set of graded skew-symmetric operations

$$\ell_n : L^{\wedge n} \to L[2-n], n > 0$$

such that for every $n > 0$ and any set of homogeneous vectors $x_1, \cdots, x_n \in V$, we have

$$\sum_{k=1}^{n} (-1)^{n-k} \sum_{\sigma \in S(n,k)} \chi(\sigma) \ell_{n-k+1}\Big(\ell_k\Big(x_{\sigma(1)}, \cdots, x_{\sigma(k)}\Big), x_{\sigma(k+1)}, \cdots, x_{\sigma(n)}\Big) = 0$$

remark 3.3.3. One can not see the n-ary brackets ℓ_n from the definition of the Lie operad, but the intuition is given as follows.

Firstly, ℓ_1 is an operation of degree 1, which serves as the differential structure for the graded k-vector space L. This is because we have for $n = 1$, the identity reads $\ell_1^2 = 0$.

Then, ℓ_2 is a skew-symmetric binary operation of degree 0. This is our Lie bracket, which must be compatible with the differential structure, giving the equation for $n = 2$.

Now we arrive at the first great difference from $\mathbb{L}_\infty$-algebras and Lie algebras (or say, DGLAs), which is the trinary operation ℓ_3 of degree -1. This operation comes from the following construction. Recall that we require the Lie bracket to have the Jacobi identity, that is,

$$[[x_1, x_2], x_3] - (-1)^{\deg x_2 \deg x_3}[[x_1, x_3], x_2] + (-1)^{\deg x_1(\deg x_2 + \deg x_3)}[[x_2, x_3], x_1] = 0 : L^{\wedge 3} \to L$$

Now in the $\mathbb{L}_\infty$-setting, we can not directly speak of a morphism equal to 0 since we are working in the derived category. Instead, we can only speak of a morphism homotopic to 0. Recall that to homotope a morphism f to 0 amounts to giving a morphism g of degree -1 such that $dg = f$. Here, the trinary bracket serves as the "Jacobiator", which is the map witnissing the Jacobi identity. However, since we require the trinary bracket also to be compatible with the differential structure, it is not directly the trinary bracket serving as the Jacobiator, but instead we have

$$\begin{aligned} &[[x_1, x_2], x_3] - (-1)^{\deg x_2 \deg x_3}[[x_1, x_3], x_2] + (-1)^{\deg x_1(\deg x_2 + \deg x_3)}[[x_2, x_3], x_1] \\ = &\, d\ell_3(x_1, x_2, x_3) + \ell_3(dx_1, x_2, x_3) + \cdots \end{aligned}$$

which explains the identity for $n = 3$.

Finally, for higher n, the operations of arities up to $n-1$ conbines together to give an $(n-1)$-ary Jacobi identity for n-elements, and this serves for the n-Jacobiator.

We now present our first example of Koszul duality.

Theorem 3.3.13 ([Sch17], Proposition 1.2.152). Let $\mathfrak{g}$ be an $\mathbb{N}$-graded vector space that is degreewise finite dimensional. Write $\mathfrak{g}^*$ for the degreewise dual, also $\mathbb{N}$-graded. Then dg-algebra structures on the Grassmann algebra $\mathrm{Sym}^* \mathfrak{g}[1]$ are in canonical bijection with $\mathbb{L}_\infty$-algebra structures on $\mathfrak{g}$.

In fact, since we have the Koszul duality for Lie algebras and cocommutative coalgebras, since the structure of the operad **Comm** is much more simpler than **Lie**, people sometimes regard morphisms of $\mathbb{L}_\infty$-algebras as the dual morphisms in coalgebras, which avoids certain technical details in $\mathbb{L}_\infty$-algebra theory.

In the next chapter, we will focus on the usage of $\mathbb{L}_\infty$-algebras in geometry, and finally see some examples of $\mathbb{L}_\infty$-algebras.

4. Lie Algebras and Deformations

In this chapter, we introduce the relation between Lie algebras and deformation problems. We work only with characteristic 0 fields k in this section.

4.1. Maurer-Cartan equations

In this section, we follow [Man22] and introduce the formalism of associating an $\mathbb{L}_\infty$-algebra to a deformation functor.

We first introduce the language of deformation functors.

Definition 4.1.1. Let $\mathbf{Art}_k$ denote the category of local Artinian k-algebras with residue field k. By definition, this category is closed under fiber products.

Definition 4.1.2. Consider $\beta : B \to A$ and $C \to A$ morphisms in $\mathbf{Art}_k$. Then for any functor $F : \mathbf{Art}_k \to \mathbf{Set}$ we have an induced map $\eta : F(B \times_A C) \to F(B) \times_{F(A)} F(C)$.

We say F is a deformation functor if for β surjective we have η surjective, and for $A = k$ we have η an isomorphism.

For a natural transformation $F \to G : \mathbf{Art}_k \to \mathbf{Set}$ is called smooth if for any $B \to A$ surjective, we have $F(B) \to G(B) \times_{G(A)} F(A)$ surjective. We say F is surjective if $F \to 0$ is surjective.

Example 4.1.1. Let A be any Noetherian local k-algebra with residue field k. Then the Yoneda functor

$$h_A : \mathbf{Art} \to \mathbf{Set}, B \mapsto \operatorname{Hom}_k(A, B)$$

is a deformation functor.

Definition 4.1.3. We define the tangent space of the deformation functor F to be the set $F(k[\varepsilon]/\varepsilon^2)$.

Definition 4.1.4. A small extension in $\mathbf{Art}_k$ is a short exact sequence

$$0 \to M \to B \to A \to 0$$

where $A, B \in \mathbf{Art}_k$, M a finite dimensional k-vector space, and $\operatorname{im} M \subset B$ is annihilated by the maximal ideal $\mathfrak{m}_B$.

Definition 4.1.5. An obstruction theory for a functor $F : \mathbf{Art}_k \to \mathbf{Set}$ with values in V a k-vector space is the data for every small extension

$$e : 0 \to M \to B \to A \to 0$$

an obstruction map $v_e : F(A) \to V \otimes M$, and natural under maps of small extensions.

remark 4.1.2. This map is called an obstruction theory because it obstructs an element in $F(A)$ from lifting to $F(B)$. Namely, we have $x \in F(A)$ lifts to $F(B)$ implies $v_e(x) = 0$. We say a obstruction theory to be complete if $v_e(x) = 0$ implies the lift exists.

We now give the way to construct from a DGLA a deformation functor.

Definition 4.1.6. We define the Maurer-Cartan equation on L to be

$$dx + [x, x] = 0, x \in L^1$$

We define its set of solutions to be $\mathrm{MC}(L)$.

Construction 4.1.7. Recall for nilpotent Lie algebras N, one may define its exponential Lie group $\exp N$, given by the same underlying space as N (recall that at least over $\mathbb{R}$, we have the exponential map for nilpotent Lie algebras a diffeomorphism to the Lie group), but equipped with the BCHD product.

We let L be a DGLA, and construct the following functors:
(1) We define $\exp_L : \mathbf{Art}_k \to \mathbf{Grp}$ by $\exp_L(A) = \exp(L^0 \otimes \mathfrak{m}_A)$. Here, $L^0 \otimes \mathfrak{m}_A$ has the Lie algebra structure given by $[l_1 \otimes x_1, l_2 \otimes x_2] = [l_1, l_2] \otimes x_1 x_2$. This is nilpotent since $\mathfrak{m}_A$ is nilpotent for an Artinian ring.
(2) We define $\mathrm{MC}_L : \mathbf{Art}_k \to \mathbf{Set}$ be given by $\mathrm{MC}_L(A) = \mathrm{MC}(L \otimes \mathfrak{m}_A)$
(3) Notice that there is an action of L^0 on L given by $[l, -]$. Hence, we have an action of the group $\exp(L^0 \otimes \mathfrak{m}_A)$ on the DGLA $L \otimes \mathfrak{m}_A$, which is also functorial in A. Thus, the definition

$$\mathrm{Def}_L(A) = \mathrm{MC}_L(A) / \exp_L(A)$$

is a well defined functor $\mathbf{Art}_k \to \mathbf{Set}$.

The functors MC_L and Def_L are both deformation functors.

Construction 4.1.8. We may construct for MC_L and Def_L complete obstruction theories with values taken in $H^2(L)$.

Consider a small extension

$$e : 0 \to M \to B \to A \to 0$$

and let $x \in \mathrm{MC}_L(A)$. We define an obstruction $v_e(x) \in H^2(L) \otimes M = H^2(L \otimes M)$ as follows. Firstly, take a lift $\tilde{x} \in L \otimes \mathfrak{m}_B$ of x, and let

$$\tilde{h} = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes \mathfrak{m}_B$$

But $\tilde{h}$ actually lies in $L^2 \otimes M$ since its image in $L^2 \otimes \mathfrak{m}_A$ is 0. Moreover, we have

$$d\tilde{h} = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h, \tilde{x}] - \frac{1}{2}[[\tilde{x}, \tilde{x}], \tilde{x}]$$

Then, the first term vanishes since $M \cdot \mathfrak{m}_B = 0$, and the second term vanishes by the Jacobi identity. Thus $\tilde{h}$ gives an element in $H^2(L \otimes M)$. Moreover, it is easy to verify that the cohomology class $\tilde{h}$ does not depend on the lift chosen.

Finally, this obstruction theory pushes forward to one on Def_L. Moreover, the theory is complete for both deformation functors (which we will not prove here).

As we will see in the following few examples, common deformation functors are controlled by Lie algebras.

Example 4.1.3. Take $k = \mathbb{C}$. Let X be a complex manifold, and take $\mathcal{U} = \{U_i\}$ a Stein covering of X. Let Θ_X denote its holomorphic tangent sheaf. Then, we obtain a semi-cosimplicial diagram of Lie algebras

$$\Theta_X(\mathcal{U}) = \prod_i \Theta_X(U_i) \rightrightarrows \prod_{i,j} \Theta_X(U_{i,j}) \xrightarrow{\cdots} \cdots$$

here $U_{i,j} = U_i \cap U_j$, and so on. Then, consider $\mathrm{Tot}(\Theta_X(\mathcal{U}))$ be the totalization of the semi-cosimplicial diagram in the category of DGLAs. Then, we have $\mathrm{Def}_{\mathrm{Tot}(\Theta_X(\mathcal{U}))}$ a deformation functor. This functor admits the following geometric meaning: it sends an Artinian ring A to the set of deformations of X to $\operatorname{Spec} A$, namely, a complex manifold Y over $\operatorname{Spec} A$ with the fiber over $\mathfrak{m}_A \in \operatorname{Spec} A$ given by X.

Example 4.1.4. We can do a bit more in the above example. Let X be a complex manifold and $Z \subset X$ be a closed submanifold. Then, we let $\Theta_X(-\log Z)$ be the kernel of the morphism of sheaves $\Theta_X \to \mathcal{N}_{Z|X}$, where $\mathcal{N}_{Z|X}$ is the normal sheaf given by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_X)$, and $\mathcal{I}_Z$ is the ideal sheaf defining Z. We have again a semisimplicial diagram of Lie algebras

$$\Theta_X(-\log Z, \mathcal{U}) = \prod_i \Theta_X(-\log Z, U_i) \rightrightarrows \prod_{i,j} \Theta_X(-\log Z, U_{i,j}) \xrightarrow{\cdots} \cdots$$

Then, we have $\mathrm{Def}_{\mathrm{Tot}(\Theta_X(-\log Z, \mathcal{U}))}$ classifying the deformation of pairs (X, Z) over an Artin ring A.

In most circumstances, we only need the DGLAs to characterize deformations. However, we still need the concept of $\mathbb{L}_\infty$-algebras in the theory of deformations to make the categorical properties better. For example, one may induce a natural $\mathbb{L}_\infty$-structure on the mapping cones of DGLAs, but the DGLA structure needs more caution.

Definition 4.1.9. Let L be an $\mathbb{L}_\infty$-algebra. We define its Maurer-Cartan equation to be

$$\sum_{n>0} \frac{1}{n!}\ell_n(x, x, \cdots, x) = 0, x \in L^1$$

The subset of Maurer-Cartan elements $\mathrm{MC}(L)$ is a solution of the Maurer-Cartan equation, and we define $\mathrm{MC}_L : \mathbf{Art}_k \to \mathbf{Set}$ again by $A \mapsto \mathrm{MC}(L \otimes A)$.

Again, the tangent space of this deformation functor is $Z^1(L)$, and $H^2(L)$ gives a complete obstruction theory for MC_L.

Construction 4.1.10. We now construct Def_L for an $\mathbb{L}_\infty$-algebra L. Before we really start, we first discuss the notion of homotopy in $\mathbb{L}_\infty$-algebras.

Recall that classically, for $f, g : X \to Y$ maps of spaces, a homotopy between f and g is a map $h : X \times [0, 1] \to Y$ such that $h(0) = f$ and $h(1) = g$. Here, in $\mathbb{L}_\infty$-algebras, we have no obvious instance of the interval $[0, 1]$. However, in this case, we have $k[t, dt]$, regarded as the sheaf of functions over $T\mathbb{A}^1$ as an $\mathbb{L}_\infty$-algebra, serves as the avatar of the interval. To believe this, notice that we have two maps $L[t, dt] \to L$ given by evaluating t at respectively 0 and 1.

This time, we define for $x, y \in \mathrm{MC}_L(A)$ to be homotopy equivalent if there is an Maurer-Cartan element $\xi \in \mathrm{MC}_{L[t,dt]}(A)$ such that $\xi(0) = x$ and $\xi(1) = y$. We define $\mathrm{Def}_L : \mathbf{Art}_k \to \mathbf{Set}$ to be the quotient of MC_L under homotopy equivalences.

At first glance, the definition of MC_L for DGLAs and $\mathbb{L}_\infty$-algebras coincide, but Def_L seems to differ much. But, we have the following theorem relating the two definitions.

Lemma 4.1.11. Let L be a DGLA. Then for two elements $x_0, x_1 \in \mathrm{MC}_L(A)$, we have the two notions of equivalences coincide, that is, $x_1 = \exp(h)x_0$ iff x_0 and x_1 are homotopy equivalent.

Proof: We prove this lemma informally. Suppose we have $x_0 = \exp(h)x_1$ for some $h \in L^0 \otimes A$. Then, consider the element $x(t) = \exp(th)x_0 \in \mathrm{MC}_{L[t,dt]}(A)$. This element clearly provides the homotopy between x_0 and x_1. Conversely, if there is some $\xi \in \mathrm{MC}_{L[t,dt]}(A)$ with $\xi(0) = x_0$ and $\xi(1) = x_1$, then there is some h such that $\xi(t)$ is indeed $\exp(th)x_0$, hence h satisfies $\exp(h)x_0 = x_1$ □

Actually, the deformation functors arising from $\mathbb{L}_\infty$-algebras comes from DGLAs, as shown in the following theorem.

Theorem 4.1.12 ([Pri19], Corollary 4.45, Theorem 4.55). Every deformation functor arising from $\mathbb{L}_\infty$-algebras regarded as an object in $DG_{\mathbb{Z}}\mathrm{Sp}$ (this is some deformation category) comes from a DGLA.

4.2. Formal moduli problems

We then presents the use of DGLAs in the deformation theory in spectral geometry (a version of derived algebraic geometry), following Part IV of Lurie's book [Lur18].

We first define the general formalism for deformations.

Definition 4.2.1. We define a deformation context to be a pair $\left(\mathcal{A}, \{E_\alpha\}_{\alpha \in T}\right)$, where $\mathcal{A}$ is a presentable ∞-category and E_α is a collection of objects in the category $\mathbf{Sp}(\mathcal{A}) := \mathbf{Sp} \otimes \mathcal{A}$ (this is equivalently the category of spectrum objects in $\mathcal{A}$).

Definition 4.2.2. We say a morphism $\varphi : A' \to A$ in $\mathcal{A}$ is elementary if there is some $\alpha \in T$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"'] & * \arrow[d] \\ A' \arrow[r] & \Omega^\infty(E_\alpha) \end{tikzcd}$$

is a pullback diagram for some map $A' \to \Omega^\infty(E_\alpha)$ ($*$ is the final object of $\mathcal{A}$).

We say a map $A' \to A$ is small if it is a composition of finitely many elementary morphisms. We say A is Artinian if $A \to *$ is small. We let $\mathcal{A}^{\text{art}}$ denote the subcategory of Artinian objects.

Definition 4.2.3. Let $(\mathcal{A}, \{E_\alpha\})$ be a deformation context. We define a formal moduli problem to be a functor $X : \mathcal{A}^{\text{art}} \to \mathbf{Ani}$ such that $X(*)$ is contractible, and satisfies the property of commuting with pullbacks $X(A \times_B B') = X(A) \times_{X(B)} X(B')$ when $B' \to B$ small.

We let $\text{Moduli}^{\mathcal{A}}$ denote the full subcategory of $\mathbf{Fun}(\mathcal{A}, \mathbf{Ani})$ spanned by the moduli problems.

Construction 4.2.4. There is a natural source formal moduli problems. Consider for any $A \in \mathcal{A}$ the functor $B \mapsto \text{Hom}_{\mathcal{A}}(A, B)$. This provides a functor $\text{Spf} : \mathcal{A}^{\text{op}} \to \text{Moduli}^{\mathcal{A}}$, called the formal spectrum functor.

Then, we consider the analogue of the tangent space of deformation functors.

Definition 4.2.5. Let $(\mathcal{A}, \{E_\alpha\})$ be a deformation context, and $X : \mathcal{A}^{\text{art}} \to \mathbf{Ani}$ be a formal moduli problem. For each α, we define the tangent space of X at α to be $Y(\Omega^\infty E_\alpha) \in \mathbf{Ani}$. Moreover, this object lifts to $X(E_\alpha) \in \mathbf{Sp}$, called the tangent complex of X at α.

Tangent complices are useful since moduli problems are essentially determined by them. Namely, we have the theorem

Theorem 4.2.6. Let $u : X \to Y \in \mathrm{Moduli}^{\mathcal{A}}$ be a morphism of formal moduli problems. Then u is an equivalence iff it induces an equivalence of tangent complices at each α.

Proof: The idea is that the Artinian objects are "generated" by the $\Omega^i E_\alpha$'s. $\square$

Now, we wish to identify the category $\mathrm{Moduli}^{\mathcal{A}}$ among one with a functor from $\mathcal{A}^{\mathrm{op}}$. This is done by the following definition.

Definition 4.2.7. We define a weak deformation theory for a deformation context $(\mathcal{A}, \{E_\alpha\})$ to be a functor $\mathfrak{D} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ such that
(1) $\mathcal{B}$ is presentable.
(2) $\mathfrak{D}$ has a right adjoint $\mathfrak{D}' : \mathcal{B} \to \mathcal{A}^{\mathrm{op}}$.
(3) There is a full subcategory $\mathcal{B}_0 \subset \mathcal{B}$ such that for every $K \in \mathcal{B}_0$, the unit $K \to \mathfrak{D}\mathfrak{D}'K$ is an equivalence.

Here, $\mathcal{B}_0$ is moreover required to be contain certain objects and closed under the following constructions
(1) The initial object of $\mathcal{B}$ $\emptyset$ is in $\mathcal{B}_0$.
(2) For every $\alpha \in T$ and every $n \geqslant 1$, there is an object $K_{\alpha,n} \in \mathcal{B}_0$ such that $\mathfrak{D}'(K_{\alpha,n}) \simeq \Omega^{\infty-n} E_\alpha$.
(3) Consider the map $v_{\alpha,n} : K_{\alpha,n} \to \emptyset$ given by applying $\mathfrak{D}$ to the basepoint $* \to \Omega^{\infty-n} E_\alpha$. Then, for a pushout square

$$\begin{tikzcd} K_{\alpha,n} \arrow[r] \arrow[d, "v_{\alpha,n}"'] & K \arrow[d] \\ \emptyset \arrow[r] & K' \end{tikzcd}$$

if $K \in \mathcal{B}_0$, so is K'.

In particular, the category of formal moduli problem equipped with the formal spectrum functor gives a weak deformation theory. In fact, the formal moduli problem is the couniversal weak deformation theory, given by the following

Construction 4.2.8. Let $\mathfrak{D} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ denote a weak deformation theory. Then for each $B \in \mathcal{B}$, consider the composition

$$\mathcal{A}^{\mathrm{art}} \hookrightarrow \mathcal{A} \xrightarrow{\mathfrak{D}^{\mathrm{op}}} \mathcal{B}^{\mathrm{op}} \xrightarrow{j_B} \mathbf{Ani}$$

where $j_B : \mathcal{B}^{\mathrm{op}} \to \mathbf{Ani}$ is the Yoneda embedding of B. This gives a functor $\Psi : \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$ such that $\Psi \circ \mathfrak{D} = \mathrm{Spf}$.

Thus, we may impose additional conditions on the functor $\mathfrak{D} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$, which would guarantee that the induced functor $\Psi : \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$ is an equivalence of categories. This gives the definition of deformation theories.

Definition 4.2.9. We first let $e_\alpha : \mathcal{B} \to \mathbf{Sp}$ be given by sending an object $B \in \mathcal{B}$ to the pushforward of E_α along $\mathcal{A} \to \mathcal{B}^{\mathrm{op}} \to \mathbf{Ani}$.

A weak deformation theory $\mathfrak{D} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ is a deformation theory if the following additional condition is met: for each α, e_α preserves filtered colimits and conservative.

Theorem 4.2.10 ([Lur18], Theorem 12.3.3.5). For any deformation theory $\mathfrak{D} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$, the functor $\Psi : \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$ is an equivalence of categories.

Now we apply the above formalism to our setting.

Construction 4.2.11. Let κ be an $\mathbb{E}_\infty$-ring, and let $\mathcal{A} = \mathbf{CAlg}_\kappa^{\mathrm{aug}}$ be the category of augmented $\mathbb{E}_\infty$-algebras over κ. Then, we have $\mathbf{Sp}(\mathcal{A}) \simeq \mathbf{Mod}_\kappa$, and let E denote the object corresponding to κ. $(\mathcal{A}, E)$ will be our deformation context. Note that this E can be regarded as the spectrum object of $\mathcal{A}$ with its n-th place the square-zero extension $\kappa \oplus \Sigma^n \kappa$.

The Artinian objects in $\mathcal{A}$ would then be objects A such that $\pi_n A = 0$ for $n < 0$ and $n \gg 0$, for each n, $\pi_n A$ is finite dimensional, and $\pi_0 A$ local. This is the derived analogue of an Artin local ring.

For $\kappa = k$ a field, the moduli problems are the functors $X : \mathcal{A} \to \mathbf{Ani}$ such that $X(k) = *$, and preserves pullbacks where $R_0 \to R_{01} \leftarrow R_1$ are surjective on π_0. We denote the category of formal moduli problems shortly by Moduli_κ in this case.

The main goal is the following:

Theorem 4.2.12 ([Lur18], Theorem 13.0.0.2). Let κ be a field of characteristic 0. Then there is a ∞-category of DGLAs over κ $\mathbf{Lie}_\kappa$, where we have a Koszul duality functor $\mathfrak{D}$: $(\mathbf{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathbf{Lie}_\kappa$ which is a deformation theory. Hence, we have an equivalence of categories $\Psi : \mathbf{Lie}_\kappa \to \mathrm{Moduli}_\kappa$.

To do this, we first define a ∞-category structure on $\mathbf{Lie}_\kappa$. To do this, we need the language of model categories. We will not introduce model categories in this essay, and the readers may find references on their own.

Construction 4.2.13. We let $\mathbf{Lie}_\kappa^{\mathrm{dg}}$ denote the model category with underlying category given by the DGLAs over κ. We equip it with the model structure given by:

(1) The weak equivalences morphisms of DGLAs that is an quasi-isomorphism on the level of chain complexes.
(2) The fibrations morphisms of DGLAs that are surjective termwise.
(3) The cofibrations morphisms having the left lifting property with respect to the trivial fibrations.

It is difficult to verify that this is indeed a model structure, so we omit the proof.

Then, we let $\mathbf{Lie}_\kappa$ denote the underlying ∞-category of $\mathbf{Lie}_\kappa^{\mathrm{dg}}$, which is the universal ∞-category with a functor from $\mathbf{Lie}_\kappa^{\mathrm{dg}}$, such that $\mathrm{Fun}(\mathbf{Lie}_\kappa, \mathcal{C})$ is the full subcategory of $\mathrm{Fun}\big(\mathbf{Lie}_\kappa^{\mathrm{dg}}, \mathcal{C}\big)$ (viewed as functor from a 1-category to an ∞-category) spanned by functors carrying weak equivalences to equivalences. This makes $\mathbf{Lie}_\kappa$ a presentable ∞-category.

This model structure is comptible to the one on the chain complex of κ-vector spaces, so the forgetful functor of model categories $\mathbf{Lie}_\kappa^{\mathrm{dg}} \to \mathbf{Mod}_\kappa^{\mathrm{dg}}$ induces a forgetful functor of ∞-categories $\mathbf{Lie}_\kappa \to \mathbf{Mod}_\kappa$. This functor actually preserves sifted colimits.

We now construct the Koszul duality functor, which is almost the same as the Grassmann algebra functor introduced at the end of the previous chapter. However, in the ∞-categorical settings, we must do a little bit more for the sake of dealing with homotopies of maps in model categories.

Construction 4.2.14. Let $\mathfrak{g}_*$ be a DGLA over κ. We define the cone of $\mathfrak{g}$, $\mathrm{Cn}_*(\mathfrak{g})$, to be the following DGLA. Firstly, we let $\mathrm{Cn}_n(\mathfrak{g}_*)$ be given by $\mathfrak{g}_n \oplus \mathfrak{g}_{n-1}$. We denote the elements by $x + \varepsilon y$, where $x \in \mathfrak{g}_n$, $y \in \mathfrak{g}_{n-1}$. Then, the differential is given by $d(x + \varepsilon y) = dx + y - \varepsilon dy$. Finally, the Lie bracket is given by

$$[x + \varepsilon y, x' + \varepsilon y'] = [x, x'] + \varepsilon\big([y, x'] + (-1)^{\deg x}[x, y']\big)$$

This is the mapping cone of the identity functor $\mathfrak{g}_* \to \mathfrak{g}_*$, so is a contractible DGLA.

The intuition for this construction is writing a resolution of κ as a left $U(\mathfrak{g})$-module, as we will see in the next construction.

Construction 4.2.15. Notice that there is a universal enveloping algebra functor from the 1-category of DGLAs over κ to the 1-category of differential graded algebras over κ, denoted by $U(\mathfrak{g})$.

Then, take $C_*(\mathfrak{g})$ to be the complex given by the tensor product $U(\mathrm{Cn}_*(\mathfrak{g})) \otimes_{U(\mathfrak{g})} \kappa$. This is called the homological Chevalley-Eilenburg complex of $\mathfrak{g}$. This construction preserves quasi-isomorphisms of DGLAs, so induces a functor on the underlying ∞-categories $C_* : \mathbf{Lie}_\kappa \to \mathbf{Mod}_\kappa$. Moreover, since the initial DGLA 0 is send to κ, C_* can be further enhanced to a functor $(\mathbf{Mod}_\kappa)_{\kappa/}$.

remark 4.2.1. In the above construction se see that $U(\mathrm{Cn}_*(\mathfrak{g}))$ is a flat resolution of κ, so computing the correct derived tensor product of chain complexes.

Lemma 4.2.16. $C_* : \mathbf{Lie}_\kappa \to (\mathbf{Mod}_\kappa)_{\kappa/}$ preserves colimits.

Proof: Omitted. □

Construction 4.2.17. We let $C^*(\mathfrak{g})$ denote the linear dual of $C_*(\mathfrak{g})$. This is called the cohomological Chevalley-Eilenberg complex. We may identify $\lambda \in C^{m(\mathfrak{g})}$ with the dual space of the degree n part of $\mathrm{Sym}^*_\kappa(\mathfrak{g}[1])$. As a consequence of the previous construction, we have C^* upgrades to a functor of ∞-categories

$$C^* : \mathbf{Lie}_\kappa \to (\mathbf{CAlg}_\kappa^{\mathrm{op}})_{\kappa/} \simeq (\mathbf{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}}$$

which preserves colimits.

Hence, by the adjoint functor theorem, C^* has a right adjoint $\mathfrak{D} : (\mathbf{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathbf{Lie}_\kappa$, which we must prove to be a deformation theory for the deformation context $(\mathbf{CAlg}_\kappa^{\mathrm{aug}}, E)$.

Proposition 4.2.18. Let $\mathfrak{g}_*$ be a DGLA over κ. Then if for every n we have $\mathfrak{g}_n$ finite dimensional and $\mathfrak{g}_n$ is trivial for $n \leqslant 0$, then the unit map $u : \mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g})$ is an equivalence in $\mathbf{Lie}_\kappa$.

Proof: This is essentially the Koszul duality we are dealing with in the previous chapter, so we omit the complete proof of this proposition. □

Corollary 4.2.19. $\mathfrak{D} : (\mathbf{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathbf{Lie}_\kappa$ is a weak deformation theory.

Proof: The first two properties is already done, so we consider the third. Consider the subcategory $\mathcal{C} \subset \mathbf{Lie}_\kappa$ spanned by the objects $\mathfrak{g}_*$ that satisfy the following conditions.
(1) $\mathfrak{g}_*$ is cofibrant.
(2) There is a finitely generated subspace $V_n \subset \mathfrak{g}_n$, trivial on non-negative degrees, such that V_* generates $\mathfrak{g}_*$ as a graded Lie algebra.

The only thing left to verify is the existence of objects K_n (in our setting the T indexing $\{E_\alpha\}_{\alpha\in T}$ is the set with one element), and the closed-under-pushout property. To show $\kappa \oplus \Sigma^n\kappa \in \mathbf{CAlg}_\kappa^{\mathrm{aug}}$ is equivalent to some $C^*(\mathfrak{g})$, notice that we may take $\mathfrak{g}$ to be the DGLA freely generated by $\kappa[-n-1]$. The closed-under-pushout property involves model categorical facts, so we omit the proof. □

Theorem 4.2.20. The functor $\mathfrak{D} : (\mathbf{CAlg}_\kappa^{\mathrm{aug}})^{\mathrm{op}} \to \mathbf{Lie}_\kappa$ is a deformation theory.

Proof: By the previous corollary, notice that the functor $e : \mathbf{Lie}_\kappa \to \mathbf{Sp}$ is given by ΣO, where O is the forgetful functor $\mathbf{Lie}_\kappa \hookrightarrow \mathbf{Mod}_\kappa \hookrightarrow \mathbf{Sp}$, and Σ is the suspension. This is because $\mathfrak{D}(E)$ is the spectrum object $\{F(\kappa[-n-1])\}$ (F denotes the free functor $\mathbf{Mod}_\kappa \to \mathbf{Lie}_\kappa$). So the conservativity and commutativity with colimits follows. □

Now we have proven the main theorem. What's more, the above arguments actually applies to $\mathbb{E}_n$-algebras!

Construction 4.2.21. Let κ be a field (not necessarily of characteristic 0). We let $\mathbf{Alg}_{\kappa}^{(n)}$ denote the $\mathbb{E}_n$-algebras in $\mathbf{Mod}_{\kappa}$, and $\mathbf{Alg}_{\kappa}^{(n),\mathrm{aug}}$ denote the augmented $\mathbb{E}_n$-algebras. Again, we may define a deformation context $\left(\mathbf{Alg}_{\kappa}^{(n),\mathrm{aug}}, E\right)$, where E is likewise defined as in the previous case. We denote the category of formal moduli problems $\mathrm{Moduli}_{\kappa}^{(n)}$.

Theorem 4.2.22 ([Lur18], Theorem 15.0.0.9). The Koszul duality functor

$$\mathfrak{D} : \left(\mathbf{Alg}_{\kappa}^{(n),\mathrm{aug}}\right)^{\mathrm{op}} \to \mathbf{Alg}_{\kappa}^{(n),\mathrm{aug}}$$

is a deformation theory, so $\Psi : \mathbf{Alg}_{\kappa}^{(n),\mathrm{aug}} \to \mathrm{Moduli}_{\kappa}^{(n)}$ is an equivalence of categories.

This more or less sketches the importance of Koszul duality in deformation theory.

Bibliography

[Chi05] M. Ching, “Bar constructions for topological operads and the Goodwillie derivatives of the identity,” Dec. 13, 2005.

[MW07] I. Moerdijk and I. Weiss, “Dendroidal sets,” Jan. 10, 2007.

[Lur17] J. Lurie, *Higher algebra*. 2017. [Online]. Available: https://www.math.ias.edu/~lurie/papers/HA.pdf

[Lur08] J. Lurie, *Higher topos theory*. 2008. [Online]. Available: https://www.math.ias.edu/~lurie/papers/HTT.pdf

[HM22] G. Heuts and I. Moerdijk, *Simplicial and dendroidal homotopy theory*. 2022.

[HM24] V. Hinich and I. Moerdijk, “On the equivalence of the Lurie's infinity operads and dendroidal infinity operads,” Oct. 09, 2024.

[Goo03] T. Goodwillie, “Calculus III: taylor series,” Oct. 30, 2003.

[Qui68] D. Quillen, “Rational homotopy theory,” June 05, 1968.

[GK94] V. Ginzburg and M. Kapranov, “Koszul duality for operads,” Oct. 1994.

[Gij24] H. Gijs, “Koszul duality and a conjecture of Francis-Gaitsgory,” Aug. 12, 2024.

[Bre95] Brenda Johnson, “The derivatives of homotopy theory,” Apr. 1995.

[CS22] M. Ching and P. Salvatore, “Koszul duality for topological En operads,” Jan. 28, 2022.

[Sch17] U. Schreiber, *Differential cohomology in a cohesive infinity topos*. 2017. [Online]. Available: https://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

[Man22] M. Manetti, *Lie methods in deformation theory*. 2022.

[Pri19] J. Pridham, “Unifying derived deformation theories,” Sept. 05, 2019.

[Lur18] J. Lurie, *Spectral algebraic geometry*. 2018. [Online]. Available: https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf